Qualifying Exam for Graduate Students

Brigham Young University Department of Physics and Astronomy

January 2022

Worked Problem Section

Instructions: In this section of the qualifying exam, you will work out your solutions to the problems you choose. Of the 14 problems, you must choose eight to answer. The eight problems you choose will be weighted equally. If you work on more than eight of the problems, please indicate clearly which eight you would like to be graded.

This section is worth 2/3 of the total exam.

The 14 problems are organized according to the following topics:

1. Mathematical Physics 1
2. Mathematical Physics 2
3. Mechanics
4. Thermodynamics
5. Electrodynamics 1
6. Electrodynamics 2
7. Quantum Mechanics 1
8. Quantum Mechanics 2
9. Optics
10. Acoustics 1
11. Acoustics 2
12. Astronomy 1
13. Astronomy 2
14. Solid State

Work each problem on the paper that has been provided. Start each problem on a new piece of paper. When you finish the exam, make sure that all of your work is placed in the appropriate divider sections. You will have four hours for this section. Student calculators are permitted.

Some possibly helpful electricity and magnetism equations:

$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ $\quad \nabla \cdot \mathbf{B} = 0$ $\quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $\quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

$\nabla \cdot \mathbf{D} = \rho_f$ $\quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$ $\quad \mathbf{D} = \epsilon \mathbf{E}$ $\quad \mathbf{P} \cdot \hat{\mathbf{n}} = \sigma_b$

$\nabla \cdot \mathbf{P} = \rho_b$ $\quad \mathbf{M} = \chi_m \mathbf{H}$

$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ $\quad \nabla \times \mathbf{M} = \mathbf{J}_b$ $\quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$ Linear $\quad \mathbf{M} \times \hat{\mathbf{n}} = \mathbf{K}_b$

Divergence Theorem

$\int \nabla \cdot \mathbf{F} d\tau = \oint \mathbf{F} \cdot d\mathbf{a}$

Stokes' Theorem

$\int (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint \mathbf{F} \cdot dl$

Name:____

MATHEMATICAL PHYSICS 1

We have two concentric spheres, the inner one of radius a and the outer one of radius b. The inner sphere has an electric potential on the surface of $u(a,\theta) = \sin^2\theta$. The outer sphere has a potential on the surface of $u(b,\theta) = 0$. In charge-free regions (i.e. anywhere that is not actually on the spheres) the electric potential obeys Laplace's equation $\nabla^2 u = 0$. Do not worry about what the spheres are made of (conductor or not), just assume that the potential on the spheres is what is specified and leave it at that.

(a) Explain why we don't have to worry about φ variation in this problem.
(b) Solve for the potential u in the region $0 \le r < a$.
(c) Solve for the potential u in the region $a < r \le b$.

For your reference the first 5 Legendre Polynomials are:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

MATHEMATICAL PHYSICS 2

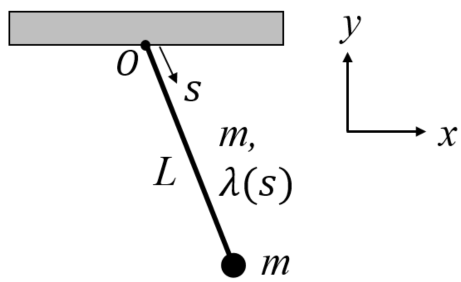
a. Express the function $f_1(x) = \cos x$ over the interval $0 < x < \pi$ as a sine Fourier series, i.e., as a Fourier series containing sine terms only. Draw figures to illustrate graphically how you proceed and define your domains carefully.

b. Use Parseval's theorem on your result under a) to show the convergence of the following series:

$$\frac{2^2}{3^2} + \frac{4^2}{(15)^2} + \frac{6^2}{(35)^2} + \cdots = \frac{\pi^2}{16}.$$

MECHANICS

A pendulum consists of a thin, rigid rod of length L and total mass m with a small bob also of mass m at the end, as shown below. The pendulum rod has a linear mass density given by $\lambda(s) = \frac{3m}{2L}(1 - (s/L)^2)$, where s is the position along the rod measured from the pivot point. The pendulum oscillates in the same vertical plane at all times.



(A) Calculate the moment of inertia of the pendulum (rod + bob) about the pivot point (labeled point O in the figure).
(B) Find the distance from the pivot point to the center of mass of the pendulum (rod + bob).
(C) Calculate the frequency of small oscillations about the pivot point.

THERMODYNAMICS

A sealed cylinder containing $n = 1$ mole of monatomic helium gas has one sliding end cap (i.e. a frictionless and massless piston), which allows the volume of the cylinder to be adjusted without any gas leakage. Imagine that the cylinder is floating in outer space, so as to be thermally, mechanically, and chemically isolated, and that it has such a shiny outer surface that it doesn't even emit black-body radiation. In other words, its total internal energy cannot change.

For a brief moment, the piston position is allowed to slide freely (i.e. with no external opposition), during which time, the cylinder volume expands from an initial volume of $V_i = 0.25\ \mathrm{m}^3$ to a final volume of $V_f = 0.50\ \mathrm{m}^3$. The final temperature of the gas is $T_f = 300$ K.

Next, the cylinder is brought inside a passing spacecraft, where the ambient temperature also happens to be 300 K. By very slowly pressing on the piston, the captain of the spacecraft isothermally and reversibly compresses the cylinder of gas back to its initial volume of $V_i = 0.25\ \mathrm{m}^3$.

Answer each question below and explain your answer. You will be scored on both the correctness of the result and the validity of the physical argument given.

a) Determine the total change in entropy (in J/K units) of the gas (final – initial) during the free expansion process.
b) Determine the total change in entropy (in J/K units) of the gas (final – initial) during the reversible compression process.
c) Determine the total change in the entropy of the combined system (consisting of the cylinder of gas and the spacecraft) resulting from the sum of the two processes.

Hints: Two well-known equations of state for a monatomic ideal gas are: $PV = nRT$ and $U = \frac{3}{2}nRT$. After solving for P and T in terms of U, V and n, one can integrate $dS = \frac{1}{T}dU + \frac{P}{T}dV$ to obtain the change in entropy. Consider the first and second laws of thermodynamics, and recall that the differential work done on the system and the heat added to the system during a reversible (but not irreversible) process are defined respectively as $dW = -PdV$ and $dQ = SdT$.

ELECTRODYNAMICS 1

A solid sphere of radius R is concentric with a conducting spherical shell that carries charge $+q_{shell}$ and has an inner radius of $2R$ and outer radius $3R$. If the electrostatic potential at the common center of the sphere and the shell is the same as the potential at infinity, what is the charge q_{sphere} on the solid sphere:
(a) if the sphere is conducting; and
(b) if the sphere has a uniform charge distribution throughout it?
(c) In the case of the uniform charge distribution throughout the sphere, draw a graph of the magnitude of the electric field vs. radius from $r = 0$ to $9R$.

ELECTRODYNAMICS 2

A spaceship passes by Earth at a speed of $0.6c$. 1 year later, according to the spaceship, they send a signal traveling at c back to the Earth. For simplicity in this problem please use units of years for time and light years for distance (which means that the magnitude of $c = 1$).

(a) Draw two space-time diagrams of the situation, one from the Earth's frame of reference and one from the spaceship's frame. Clearly label the world lines of the spaceship and the Earth, as well as the "signal sent" and "signal received" events and of course the signal itself.
(b) Determine the space-time coordinates of the "signal sent" and "signal received" events in both frames of reference.
(c) Specifically, how much distance did the signal travel and how long did it take to travel the distance, in each frame of reference?

Here are the Lorentz transformation equations, with frame 2 moving to the right with respect to frame 1:

$$\begin{pmatrix} x \\ ct \end{pmatrix}_{frame\ 1} = \begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}_{frame\ 2}$$

(If frame 2 is moving to the left with respect to frame 1, then negate the off-diagonal elements.)

QUANTUM MECHANICS 1

Commutation relations

From Wikipedia we have:

In quantum mechanics, the momentum operator is the operator associated with the linear momentum. The momentum operator is, in the position representation, an example of a differential operator. For the case of one particle in one spatial dimension, the definition is:

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

where $\hbar$ is Planck's reduced constant $\hbar = \frac{h}{2\pi}$, i the imaginary unit, x is the spatial coordinate, and a partial derivative (denoted by $\frac{\partial}{\partial x}$) is used instead of a total derivative ($\frac{d}{dx}$) since the wave function is also a function of time. The "hat;; indicates an operator. The "application" of the operator on a differentiable wave function is as follows:

$$\hat{p}\psi = -i\hbar \frac{\partial \psi}{\partial x}$$

Assume ψ is a Gaussian. That is, $\psi(x,t) = Ae^{-(kx)^2}$. Also A is a normalizing constant, and k has units of reciprocal length.

a Evaluate this expression for the given ψ. Yes, do the derivative.

$$\hat{p}\psi = -i\hbar \frac{\partial \psi}{\partial x}$$

b Write down what $x\hat{p}\psi$ is. That is, take the result from part a and multiply it by x on the left hand side.

c Now evaluate $\hat{p}x\psi$. That is, use the operator $\hat{p}$ on the product of x and ψ. Do the derivative.

d Now compute the commutation relationship between x and operator $\hat{p}$ using ψ. That is, $[x,\hat{p}]\psi = (x\hat{p} - \hat{p}x)\psi$. Show that the answer will be $i\hbar\psi$. In other words, show that $[x,\hat{p}] = i\hbar$.

e Using the same method as part d, compute the commutation relationship for $[x^n,\hat{p}]$.

f For any function $f(x)$ that can be expanded in a power series, show that $[f(x),\hat{p}] = i\hbar f'(x)$.

QUANTUM MECHANICS 2

Emission – absorption in two level systems

Consider a two-level system of energy levels E_a and E_b, in which the populations are N_a and N_b respectively. The system is perturbed by an electromagnetic radiation of energy density ρ.

a) Name all the possible transition processes that can occur between the two levels. Represent these transitions on an energy diagram. Introduce the Einstein coefficients *A* (spontaneous) and *B* (stimulated), respectively.

b) Consider the spontaneous emission only from level E_b down to E_a. Write an equation for $\frac{dN_b}{dt}$ using the coefficient *A* and N_b. Solve for $N_b(t)$. Define the 'half-time' in terms of *A*.

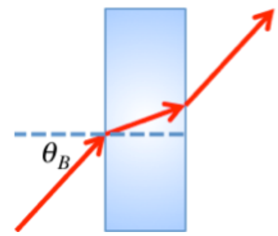
c) Now consider all the possible transition processes. Write an equation for $\frac{dN_b}{dt}$ in terms of the coefficients *A* and *B*, $N_{a,}$ N_b and ρ.

d) Express the condition for thermal equilibrium and a resulting relationship between *A* and *B*, $N_{a,}$ N_b and ρ.

Hint: Like for radioactivity, rates of change $\frac{dN}{dt}$ are directly proportional to some population number N (use the proper one). For stimulated processes, it is also proportional to the energy density of the radiation. Remember that $\frac{dN}{dt}$ is positive when the energy level is being replenished and negative when the energy level is being depleted

OPTICS

A thick window is installed at Brewster's angle in a narrow p-polarized laser beam such that it transmits with $T = 100\%$.



(**a**) *Derive* Brewster's angle and compute it for $n = 1.5$.

(b) If s-polarized light is sent through the window instead of p-polarized light, what fraction of the original power will go all the way through? Neglect multiple reflections or interferences within the window.

Fresnel coefficients:

$$r_s \equiv \frac{E_s^{(r)}}{E_s^{(i)}} = \frac{\sin\theta_t\cos\theta_i - \sin\theta_i\cos\theta_t}{\sin\theta_t\cos\theta_i + \sin\theta_i\cos\theta_t} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}, \quad t_s \equiv \frac{E_s^{(t)}}{E_s^{(i)}} = \frac{2\sin\theta_t\cos\theta_i}{\sin\theta_t\cos\theta_i + \sin\theta_i\cos\theta_t} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)}$$

$$r_p \equiv \frac{E_p^{(r)}}{E_p^{(i)}} = \frac{\cos\theta_t\sin\theta_t - \cos\theta_i\sin\theta_i}{\cos\theta_t\sin\theta_t + \cos\theta_i\sin\theta_i} = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}, \quad t_p \equiv \frac{E_p^{(t)}}{E_p^{(i)}} = \frac{2\cos\theta_i\sin\theta_t}{\cos\theta_t\sin\theta_t + \cos\theta_i\sin\theta_i} = \frac{2\cos\theta_i\sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

ACOUSTICS 1

Consider a point source in a free field that radiates with a frequency of 500 Hz with a sound pressure level of 110 dB re 20 µPa at 1 m.

(A) What is the acoustic intensity at r = 8 m?
(B) What is the sound power level produced by the source?
(C) What is the amplitude and phase (relative to the acoustic pressure) of the acoustic particle velocity at r = 1 m and r = 8 m?
(D) As the phase approaches zero, what is happening to the wavefronts?
(E) Consider a second source, located 1m from the first source, with equal source strength and frequency. What is the combined sound pressure level at a microphone located 5 m from each source?

ACOUSTICS 2

A plane wave in water of 100 Pa peak pressure amplitude is incident at 30° on a mud bottom having $\rho_2 = 2000$ kg/m^3 and $c_2 = 1000$ m/s. Compute

a) the angle of the ray transmitted into the mud,
b) the peak pressure amplitude of the transmitted ray,
c) the peak pressure amplitude of the reflected ray,
d) the sound power reflection coefficient, and
e) the smallest angle of incidence at which all of the incident energy will be reflected.

(Take the properties of the water to be $\rho_1 = 1000$ kg/m^3 and $c_1 = 1500$ m/s.)

ASTRONOMY 1

a. Discuss the meaning of thermodynamic equilibrium. Is this a good model for stellar interiors? (Why or why not?) Now what assumption do we make in stellar interiors that allows us to almost use thermodynamic equilibrium? And how do we test this at any given point in the star? Where does it breakdown?

b. Let's think about three objects; 1) a sphere with the radius of the Sun, 2) a cube with each side 2x the solar radius, and 3) a flattened disk with a radius of one solar radii. Let's assume a temperature in all cases of 5800 K and are emitting as blackbodies. Do the following:
    i. Determine the luminosity of each object.
    ii. Which of the models would you consider representative of a real object and why?
    iii. As you move away from the objects, would any of the objects show a different drop in flux?
    iv. What kind of object is #3 (the flattened disk)? What would make this a more realistic model?

Potentially useful data:
L-solar = 3.846×10^{26} Watts   R-solar = 695500 km $\sigma = 5.670373 \times 10^{-8}$ W/m^2/K^4

ASTRONOMY 2

Consider the case of a globular cluster that forms $N_{\text{stars}} = 2 \times 10^5$ stars out of a large gas cloud with a mass of $M_{\text{gas}} = 10^{5.5}\ M_\odot$. Assume that as stars begin to form, their zero-age Main Sequence (ZAMS) masses follow a Kroupa initial mass function (IMF):

$$\xi_{\text{KR}}(M) = \begin{cases} \xi_0' \left(\frac{M}{M_\odot}\right)^{-0.3} & \text{for } (M/M_\odot) < 0.08 \\ \xi_0'' \left(\frac{M}{M_\odot}\right)^{-1.3} & \text{for } 0.08 \leq (M/M_\odot) < 0.5 \\ \xi_0''' \left(\frac{M}{M_\odot}\right)^{-2.3} & \text{for } (M/M_\odot) > 0.5 \end{cases}$$

where $dN_{\text{stars}} = \xi(M)dM$ is the number of stars formed with masses between M and $M + dM$, and each respective ξ_0^n (in stars $M_\odot^{-1}$) is needed to both scale $\xi_{\text{KR}}(M)$ and to make it piecewise continuous.

(a) Integrate the above IMF over the appropriate mass ranges to calculate ξ_0', ξ_0'', and ξ_0'''. You can ignore brown dwarfs. Make a reasonable assumption for the minimum (M_{min}) and maximum (M_{max}) stellar masses **and justify your choice.** Note that working in $M_\odot$ units is easiest.

(b) For the Kroupa IMF, find the total stellar mass

$$M_{\text{stars}} = \int_{M_{\text{min}}}^{M_{\text{max}}} M'\xi(M')dM'$$

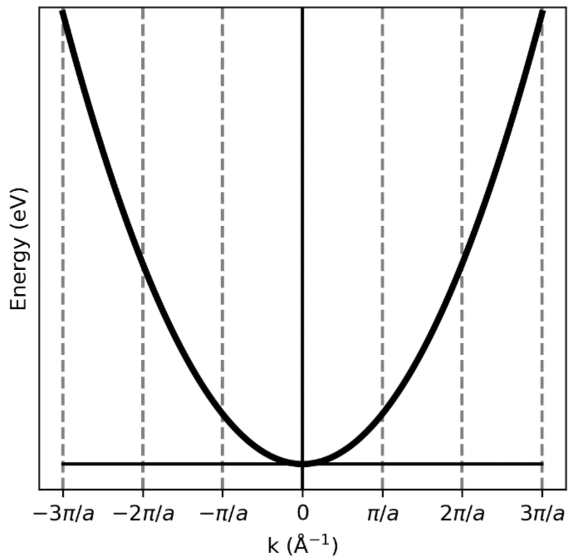
For this IMF, what is the efficiency of the star formation process (i.e., what is $M_{\text{stars}}/M_{\text{gas}}$)?

(c) Determine the mean stellar mass $\langle m_{\text{star}} \rangle$ for this IMF.

(d) Determine the number of black holes that are expected to form. You may assume a black hole will form from any star with a minimum ZAMS mass of 25 $M_\odot$ up to M_{max} you selected in part (a).

SOLID STATE

Consider a one-dimensional (1D) crystal consisting of identical atoms placed a distance a apart from each other along an infinite line. In the free electron model of this solid, the energy dispersion relation has a quadratic dependence on the crystal momentum $\vec{k}$ of the electron. The band structure for this 1D crystal is shown in the extended Brillouin zone scheme below.



(A) Still within the free electron model of this crystal, sketch the band structure in the reduced Brillouin zone scheme (i.e., between $-\frac{\pi}{a}$ and $\frac{\pi}{a}$). Briefly explain or illustrate how you constructed this diagram from the previous diagram shown in the extended zone scheme.

(B) Now we turn on a small potential energy function with the same spatial periodicity as the crystal lattice. Sketch the band structure and briefly explain how it differs from part (A).

(C) Using your band structure diagram from part (B), briefly explain the difference in electrical conductivity that would be observed if each atom has one electron compared to each atom having two electrons. In which case is the electrical conductivity higher? Assume there are no distortions to the lattice.

(D) The Peierls transition is a structural distortion of a 1D lattice where the atoms form dimers, which results in a doubling of the lattice spacing (see diagram below). Using band structure and energy arguments, briefly explain why the Peierls transition would be expected to occur spontaneously in a 1D crystal where each atom has one electron.