Qualifying Exam for Graduate Students

Brigham Young University Department of Physics and Astronomy

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Worked Problem Section

Instructions: In this section of the qualifying exam, you will work out your solutions to the problems you choose. Of the 14 problems, you must choose eight to answer. The eight problems you choose will be weighted equally. If you work on more than eight of the problems, please indicate clearly which eight you would like to be graded.

This section is worth 2/3 of the total exam.

The 14 problems are organized according to the following topics:

1. Mathematical Physics 1
2. Mathematical Physics 2
3. Mechanics
4. Thermodynamics
5. Electrodynamics 1
6. Electrodynamics 2
7. Quantum Mechanics 1
8. Quantum Mechanics 2
9. Optics
10. Acoustics 1
11. Acoustics 2
12. Astronomy 1
13. Astronomy 2
14. Solid State

Work each problem on the paper that has been provided. Start each problem on a new piece of paper. When you finish the exam, make sure that all of your work is placed in the appropriate divider sections. You will have four hours for this section. Student calculators are permitted.

Some possibly helpful electricity and magnetism equations:

$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\nabla \cdot \mathbf{B} = 0$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$
$\nabla \cdot \mathbf{D} = \rho_f$	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\mathbf{D} = \epsilon \mathbf{E}$	$\mathbf{P} \cdot \hat{\mathbf{n}} = \sigma_b$
$\nabla \cdot \mathbf{P} = \rho_b$		$\mathbf{M} = \chi_m \mathbf{H}$	
$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$	$\nabla \times \mathbf{M} = \mathbf{J}_b$	$\mathbf{H} = \frac{1}{\mu} \mathbf{B}$ Linear	$\mathbf{M} \times \hat{\mathbf{n}} = \mathbf{K}_b$

Divergence Theorem

$\int \nabla \cdot \mathbf{F} d\tau = \oint \mathbf{F} \cdot d\mathbf{a}$

Stokes' Theorem

$\int (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint \mathbf{F} \cdot dl$

Name:____

MATHEMATICAL PHYSICS 1

(a) We have a bar of length π that has been sitting in a boiling bath of water at 100° for a long time. At $t = 0$ we attach the bar to two fixed temperature reservoirs, so each end is stuck at a particular temperature. The reservoirs are such that $u(0, t) = 100$ and $u(\pi, t) = 10$. Find the temperature as a function of x and t, $u(x, t)$, for $t > 0$. The heat equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

(b) Sketch what the solution looks like at a moderately long time, i.e. long enough that it no longer looks like the initial temperature, but short enough that it has not gone to its steady state completely.

MATHEMATICAL PHYSICS 2

a. Consider the function $f(x) = (\pi - x)\sin x$, defined on the domain $-\pi \le x \le \pi$. Sketch this function over its domain.
b. Comment on whether this function is even or odd.
c. Construct the Fourier series of this function over its domain.
d. What infinite series could be evaluated by applying Parseval's identity to this function? Just write the formal sum. No need to evaluate the expression and find to what value the sum converges.

MECHANICS

A particle of mass m exists on the x axis and is subject to a force $F = kx$, where $k > 0$ (note that this is NOT Hooke's Law because we do not have a minus sign). From Newton's 2nd Law, then, the equation of motion is $m\ddot{x} = kx$.

A) Determine the potential energy of the particle as a function of position x.
B) Write down the Lagrangian and use it to determine the equation of motion (which should be consistent with Newton's 2nd Law, of course).
C) Solve the differential equation by assuming a solution of the form e^{rt} and solving for r. Since this is a second-order differential equation, your general solution should be the sum of two terms.
D) Briefly comment on the general motion of this particle compared to that of a particle subject to Hooke's Law. In particular, is the motion of this particle bounded or unbounded?
E) Find the Hamiltonian for this particle (remember to express it as a function of some generalized momentum and coordinate).

THERMODYNAMICS

A system of N particles is in thermal contact with a reservoir of temperature T. Each particle must be in one of two discrete energy levels, a singly-degenerate ground state with energy ϵ_0 or a doubly-degenerate excited state with energy ϵ_1.

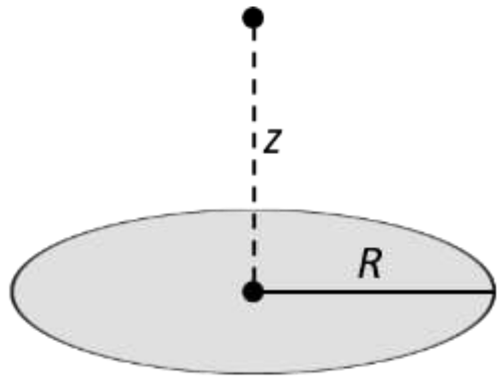
a) Using a spectroscopic technique, you discover that 2/3 of the particles are in the ground state. Express the energy-level difference $\Delta = \epsilon_1 - \epsilon_0$ in terms of kT, where k is the Boltzmann constant and T is the system temperature. Hint: Relate the single-particle partition function to the probability of occupying the ground state.

b) Calculate the average energy per particle purely in terms of ϵ_0 and ϵ_1, which should not include any factors of Δ or kT.

ELECTRODYNAMICS 1

A) Use Gauss's Law to calculate the electric field $\vec{E}$ at a point a distance z above an infinite sheet of charge of uniform charge density σ.

B) Now consider an electric charge Q distributed uniformly over a thin, circular disc of radius R, as shown below. Calculate the <u>electric potential</u> V at a point a distance z directly above the center of the disc.



C) Now let's say you want to calculate the electric field $\vec{E}$ at that same point a distance z above the center of the disc. Briefly explain why a simple application of Gauss's Law will not be very helpful for doing this. What would be a better method?

D) Using whatever method you like, calculate the electric field $\vec{E}$ at that point.

E) Your answer to part (D) should agree with your answer to part (A) in the limit that z approaches zero. Show that this is the case and briefly explain in words why this is.

ELECTRODYNAMICS 2

The Lienard-Wiechert potentials describe the scalar and vector potentials at a point $\mathbf{r}$ created by a moving point charge at location $\mathbf{r}'$, and the equations are as follows:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{v}{c^2}V(\mathbf{r},t)$$

where $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'(t_r)$, the vector from the location of the point charge *at the retarded time* t_r to the point where you want to know the field, $\mathbf{r}$.

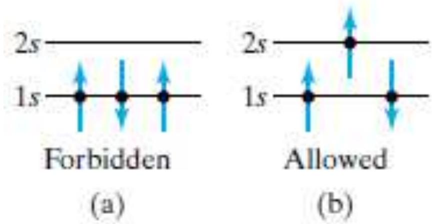
(a) In general, how can you determine the **E** and **B** fields from the potentials V and $\mathbf{A}$?

(b) A point charge q moves in a counter-clockwise circle of radius a at constant angular velocity ω. The circle lies in the x-y plane, centered at the origin, and at time $t = 0$ the charge is at position $(a, 0)$. Find the Lienard-Wiechert potentials V and A, for points on the z-axis. Hint: first determine the position of the point charge as a function of time.

QUANTUM MECHANICS 1

1. Consider the schematic for neutral lithium energy levels shown to the right. Lithium has three electrons.
   a. Why is configuration (a) forbidden?

   b. Whose name is associated with this law?



2. You will find in the two tables on the following page the energy levels of neutral hydrogen (left) and neutral lithium (right) as given in standard spectroscopy tables. Hydrogen has one proton in the nucleus while lithium has three. Here, configuration means the energy level for the outermost electron. *J* refers to the coupling of the electron's various angular momenta. You do not need to worry about the "Term" column. Treat it as a label. "Level" refers to energy levels. It is in units that you may not have seen. (1/λ) is measured in reciprocal centimeters. Reciprocal centimeter means the number of wavelengths in 1 cm. You can compute energy differences by subtracting one level from the other using the formula: $E = (1.24 \times 10^{-4}\ \text{eV} \cdot \text{cm}) \times (1/\lambda)$.

   a. Calculate the energy difference between the 3d (*J*=3/2) and "Limit" for lithium. That is, calculate the ionization energy of lithium from the 3d level.
      Energy (3d to ionization) in reciprocal centimeters: ____________cm^{-1}
      Conversion to eV:_______ eV. Note: This should be a number between 1 and 3 eV. Five decimal accuracy is satisfactory.
   b. Now repeat this same calculation for hydrogen, i.e. calculate the ionization energy from the 3d (*J*=3/2) level.
      Energy (3d to ionization) in reciprocal centimeters: ____________cm^{-1}
      Conversion to eV:_______ eV.

   c. Why is the energy difference so similar for the two cases? (answer in about 20-35 words)

   d. Why is the magnitude of the energy slightly larger in the case of lithium? (answer in about 20-35 words)

   e. Extra credit: why are the three levels, 3s, 3p, 3d identical in the case of hydrogen, but are different in the case of lithium? Think about which wave function allows the electron in the third shell to get close to the nucleus and which function keeps the electron far from the nucleus.

Energy levels of neutral Hydrogen

Configuration	Term	J	Level (cm^{-1})
1*s*	^2S	1/2	0.0000
2*p*	$^2P°$	1/2	82258.9191
		3/2	82259.2850
2*s*	^2S	1/2	82258.9544
3*p*	$^2P°$	1/2	97492.2112
		3/2	97492.3196
3*s*	^2S	1/2	97492.2217
3*d*	^2D	3/2	97492.3195
		5/2	97492.3556
4*p*	$^2P°$	1/2	102823.8486
		3/2	102823.8943
4*s*	^2S	1/2	102823.8530
4*d*	^2D	3/2	102823.8942
		5/2	102823.9095
4*f*	$^2F°$	5/2	102823.9095
		7/2	102823.9171
5*p*	$^2P°$	1/2	105291.6287
		3/2	105291.6521
5*s*	^2S	1/2	105291.6309
5*d*	^2D	3/2	105291.6520
		5/2	105291.6599
5*f*	$^2F°$	5/2	105291.6598
		7/2	105291.6637
5*g*	^2G	7/2	105291.6637
		9/2	105291.6661
H	***Limit***		**109678.7717**

Energy levels of neutral Lithium

Configuration	Term	J	Level (cm^{-1})
2*s*	^2S	1/2	0.000
2*p*	$^2P°$	1/2	14903.622
		3/2	14903.957
3*s*	^2S	1/2	27206.066
3*p*	$^2P°$	1/2	30925.517
		3/2	30925.613
3*d*	^2D	3/2	31283.018
		5/2	31283.053
4*p*	$^2P°$	1/2	36469.714
		3/2	36469.754
4*d*	^2D	3/2	36623.297
		5/2	36623.312
Li II $1s^2$ ($^1S_0$)	***Limit***		**43487.150**

QUANTUM MECHANICS 2

Consider a particle with spin s = 1/2 precessing in 3D space in the presence of a magnetic field:

$$\vec{B} = B_0\vec{k}$$

The magnetic interaction induces a perturbation in the Hamiltonian:

$$H' = -\vec{M}.\vec{B} = -\gamma\vec{S}.\vec{B}$$

where $\vec{S} = S_x\vec{\imath} + S_y\vec{\jmath} + S_z\vec{k}$ is the 3D spin vector operator and γ the gyromagnetic constant

a) Use the Pauli matrices below to express the perturbation $H'(t)$ as a matrix in the spinor space

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \; S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \; S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

b) Assume the particle's spinor state is $\chi(t) = (a(t), b(t))$ in the basis of the eigenstates of S_z.

Use the Schrödinger equation (see below) to derive two equations: one for $a(t)$ and one for $b(t)$.

c) Assuming the initial spinor state is $\chi(0) = (\cos\alpha, \; \sin\alpha)$, express $\chi(t)$ at later times.

d) Calculate the expectation value of $\langle S_x \rangle$ as a function of time.

What is the physical meaning of $\langle S_x \rangle$?

Hint: Schrödinger equation in spinor space (also known as equation of motion) $i\hbar \frac{d\chi}{dt} = H'\chi$

OPTICS

(a) Derive the ABCD matrix for propagating a paraxial ray a distance *d* along the optic axis of system. The bulk of the points will be awarded for showing the process of deriving the matrix, so simply writing the answer will not get you very far. Include a clear diagram and some sentences to describe what you are doing in the derivation.

(b) A researcher desires to make a laser cavity using a concave mirror of radius *R* and a flat mirror (infinite radius) separated by a distance *d* with a gain medium in between the mirrors. Ignoring the gain medium, write down an ABCD matrix for the roundtrip through this cavity. You may be interested to know that reflection from a mirror of radius *R* is accomplished using this matrix:

$$\begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$$

(c) James Sylvester proved the following theorem about 2x2 matrices with determinant 1, such as the one you made in (b):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin\theta}\begin{bmatrix} A\sin N\theta - \sin(N-1)\theta & B\sin N\theta \\ C\sin N\theta & D\sin N\theta - \sin(N-1)\theta \end{bmatrix}$$

where

$$\cos\theta = \frac{1}{2}(A+D)$$

Use this theorem to derive a condition for the stability of the laser cavity in terms of the parameters given above. Again, the clarity of your derivation will count more heavily than getting the right answer. Convince me that you thoroughly understand how to get from Sylvester's theorem to the cavity stability condition.

ACOUSTICS 1

Consider a rectangular membrane, of width L_x by L_y, fixed on the edges

(a) Find the lowest modal frequency (fundamental frequency).
(b) If $L_x = 2L_y$, compute the ratio of each of the first four overtones to relative to the fundamental frequency.
(c) Are any of the frequencies for any first four overtones likely to produce audible beating or roughness? If so, identify which ones and explain why.
(d) For what ratio of $L_x/\mathrm{L_y}$ are the (3,1) and (1,2) modes degenerate?

ACOUSTICS 2

Two loudspeakers are broadcasting sound simultaneously, and the sound pressure level (SPL) measured at a nearby microphone (3 m in front of loudspeaker A) is 89 dB re 20 µPa. If loudspeaker A alone produces an SPL at the microphone of 85 dB re 20 µPa,
(a) what is the SPL produced at the microphone by loudspeaker B, assuming the two sources are incoherent?

(b) If the microphone has a sensitivity of 60 mV/Pa, what is the rms voltage produced with both loudspeakers broadcasting together?

(c) Assuming the loudspeakers are radiating as monopoles, what is the intensity produced by loudspeaker A at a distance of 14 m?

(d) What is the sound power output from loudspeaker A?

(e) If instead the two loudspeakers are coherent, and exactly in-phase at the microphone, what is the SPL produced at the microphone by loudspeaker B, if loudspeaker A alone still produces 85 dB re 20 µPa at the loudspeaker?

ASTRONOMY 1

We often look at faint objects like stars and nebulae through a telescope. Describe the difference in the measurements we obtain with Earth-based telescope between a star and a nebulae. Now, let's say we had a high speed space craft and could get much closer to the star or nebulae. How would our experience of the brightness each object change as a function of distance? Show this mathematically. Does the resolution of the telescope make a difference in either case? How?

ASTRONOMY 2

Assuming a flat universe ($k = 0$), the Friedmann equations can be used to derive:

$$\dot{\rho} = -3H\left(\rho + \frac{P}{c^2}\right)$$

where ρ is the total matter-energy density, $H \equiv \dot{R}/R$ is the Hubble parameter and R is the scale factor, and P is the pressure. For a perfect fluid, the equation of state can be given by $P = \omega\rho c^2$ with dimensionless parameter ω.

(a) Derive the fluid equation $R^{3(1+\omega)}\rho = \text{constant}$, which is valid for each component ($\rho_\text{m}, \rho_\text{rad}, \rho_\Lambda$) of the perfect fluid. Show all work.

(b) From the previous equation for $\dot{\rho}$, determine the equation-of-state ω_Λ of dark energy.

(c) For the sake of this exercise, assume that radiation behaves like matter, such that its pressure is negligible (i.e., that $\omega_\text{rad} \approx \omega_\text{m}$ in the equation of state $P = \omega\rho c^2$). However, assume the radiation energy density remains $u_\text{rad} = aT^4$, with a being a constant. Recall that the fluid equation $R^{3(1+\omega)}\rho = \text{constant}$ can also be expressed in terms of the energy density u: $R^{3(1+\omega_\text{rad})}u_\text{rad} = \text{constant}$. If radiation in our universe provides negligible pressure, estimate what the current cosmic microwave background temperature would be, along with the peak wavelength of this black body emission.

(It may be helpful to recall that at the time of the Big Bang nucleosynthesis, when $R_\text{BB} \ll R_0$ for the present scale factor R_0, the production of helium occurred at a temperature of about 10^9 K and at a baryon density $\rho_\text{b,BB}$ that is about 2.5×10^{25} times greater than the present $\rho_\text{b,0}$. Also, recall that the Wien's displacement constant is b = 2898 μm K.)

SOLID STATE

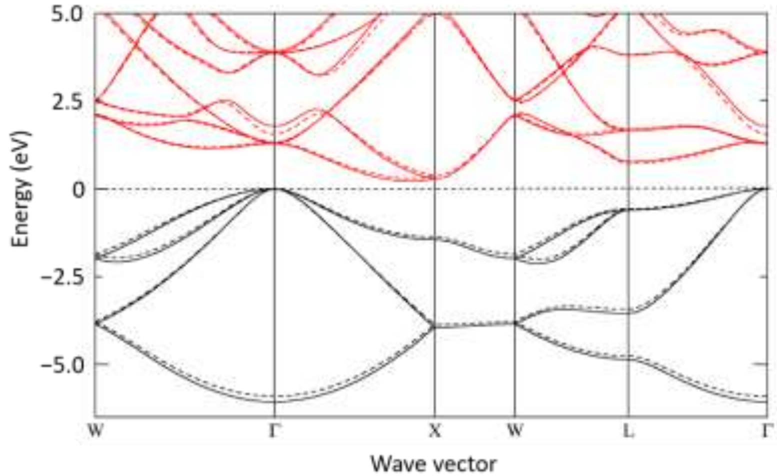
(A) Derive the electronic density of states $g(E)$, where E is energy, for the free-electron model in a three-dimensional solid.

(B) Consider the three figures of band structures shown on the next page, labeled #1, #2, and #3. One corresponds to a metal, one to an insulator, and one to a semiconductor. Classify each band structure diagram as metal, insulator, or semiconductor, and briefly justify your reasoning. Note that the Fermi level defines zero energy in each case.

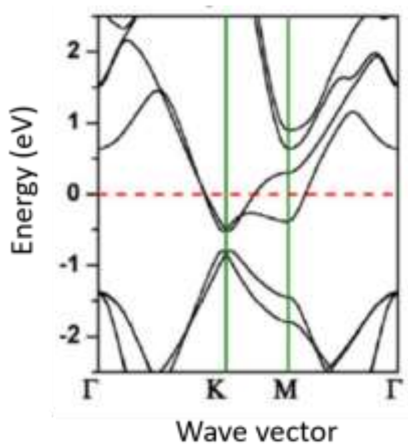
(C) Sketch the electrical resistivity as a function of temperature for a metal and then for an insulator. Briefly explain the reason for the shape of the resistivity curve in each case.

(D) Explain why, even at absolute zero temperature, some electrons can have speeds on the order of 10^6 m/s.

Band structure #1:



Band structure #2:



Band structure #3: