Physics 321

Hour 7
Energy Conservation – Potential Energy in One Dimension

Bottom Line
- Energy is conserved.
- Kinetic energy is a definite concept.
- If we can determine the kinetic energy at all points in space by knowing it at one point in space, we can invent a potential energy so that energy can be conserved.
- Kinetic energy is related to work.
- Potential energy must also be related to work.

Kinetic Energy
\[ T = \frac{1}{2}mv^2 \]
\[ P = \dot{T} = \frac{1}{2}m2v\dot{v} = Fv \]

Does that make sense?
Is it sometimes true?
Is it always true?

Work-Energy Theorem
\[ T = \frac{1}{2}\frac{\vec{p} \cdot \vec{p}}{2m} \]
\[ P = \dot{T} = \frac{\vec{p} \cdot \vec{\dot{p}}}{2m} + \frac{\vec{\dot{p}} \cdot \vec{p}}{m} = \vec{F} \cdot \vec{\dot{v}} \]

This is a useful relation – but we’ll go one step further:
\[ \Delta T = \frac{\vec{F} \cdot \Delta \vec{v}}{\Delta t} \rightarrow dT = \vec{F} \cdot d\vec{v} \]
\[ \Delta T_{12} = \int_{1}^{2} \vec{F} \cdot d\vec{r} = W_{1-2} \]
- Positive work increases kinetic energy, negative work decreases kinetic energy

Gravity
Depending on initial velocity, an object can move freely under the influence of gravity in many different paths from 1 to 2.

In each case:
\[ W = \int_{1}^{2} \vec{F} \cdot d\vec{r} = \int_{1}^{2} \vec{F}_g d\vec{y} = -mg \int_{\frac{1}{2}}^{2} dy = -mg (y_2 - y_1) = -mg\Delta y \]

And on any closed path \( W = \oint \vec{F} \cdot d\vec{r} = 0 \)

This means that \( \Delta T \) from 1 to 2 is independent of path. If we know \( T_1 \) we also know \( T_2 \).
Potential Energy

If $\Delta T$ is dependent only on the end points of a path (not on the path or on time), we can define a potential energy. Otherwise, we can not.

$$W = \int \vec{F} \cdot d\vec{r} = -mg\Delta y = -\Delta U$$

Stokes' Theorem I

$$\oint_{P_1} \vec{F} \cdot d\vec{r} = \oint_{P_2} \vec{F} \cdot d\vec{r} + \oint_{P_3} \vec{F} \cdot d\vec{r}$$

Note the path integrals on the mid-line cancel.

$$\oint_{P_1} \vec{F} \cdot d\vec{r} = \left( \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{P} \vec{F} \cdot d\vec{r} \right) dA$$

$$\oint_{P_1} \vec{F} \cdot d\vec{r} = \int \left( \text{curl } \vec{F} \right) \cdot d\vec{A}$$

Potential Energy

If the curl of $\vec{F}$ is zero and $\vec{F}$ has no explicit time dependence, we can define a potential energy. Otherwise, we can not.

In general, if $U = -\int \vec{F} \cdot d\vec{r}$, $\vec{F} = -\nabla U$.

Since $\nabla \times (\nabla U) \equiv 0$ the curl of $\vec{F}$ must be zero.

Stokes' Theorem II

$$\oint_{P_1} \vec{F} \cdot d\vec{r} = \sum_i \oint_{P_i} \vec{F} \cdot d\vec{r}$$

$$= \sum_i \left( \frac{1}{\Delta A_i} \oint_{P_i} \vec{F} \cdot d\vec{r} \right) \Delta A_i$$

$$= \int \left( \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{P} \vec{F} \cdot d\vec{r} \right) dA$$

$$\oint_{P_1} \vec{F} \cdot d\vec{r} = \int (\text{curl } \vec{F}) \cdot d\vec{A}$$

Differential Form of the Curl

$$\left( \text{curl } F \right)_x = \frac{\left[ F_y(x, y, z) + F_0(x, y, z) \Delta y \right] - \left[ F_y(x + \Delta x, y, z) + F_0(x + \Delta x, y, z) \right] \Delta y}{\Delta x \Delta y}$$

$$= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

$$\nabla \times \vec{A} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
A_x & A_y & A_z
\end{vmatrix}$$

Spherical Coordinates

It's important to go back and forth between spherical and Cartesian coordinates. Know these:

$$x = r \sin \theta \cos \varphi$$
$$y = r \sin \theta \sin \varphi$$
$$z = r \cos \theta$$

$$r^2 = x^2 + y^2 + z^2$$

$$\tan \varphi = \frac{y}{x}$$

$$\tan \theta = \sqrt{x^2 + y^2} / z$$
Cylindrical Coordinates

It’s also important to go back and forth between cylindrical and Cartesian coordinates. Know these:

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = z \]

\[ r^2 = x^2 + y^2 \]
\[ \tan \theta = \frac{y}{x} \]