Physics 321

Hour 38
Scattering Cross Sections

The Scattering Problem
• In the cm frame, two particles approach each other and scatter. Given the interaction, what is the distribution of scattered particles?

The Inverse Scattering Problem
• Given the distribution of scattered particles, what is the interaction?
• This is a very hard but very interesting problem.

The Scattering Problem
• The impact need not be head-on. The distance between the trajectories is the impact parameter, $b$.

Relating $\vartheta$ to $b$
• The first thing we need to do is find the relationship between $\vartheta$ and $b$ for a given type of interaction.

Hard Sphere Scattering $- b(\vartheta)$
\[
\cos \frac{\vartheta}{2} = \sin \left( \frac{\pi}{2} - \frac{\vartheta}{2} \right) = \frac{b}{a}
\]
\[
b = a \sin \frac{\pi - \vartheta}{2}
\]
Coulomb Scattering $- b(\vartheta)$

First we note that the total angular momentum is

$$\ell = p_a b + p_b b = pb$$

$$E = T(\infty) = \frac{p^2}{2\mu} = \frac{\ell^2}{2\mu b^2}$$

Coulomb Scattering $- b(\vartheta)$

Next we use a result from the central force problem.

$$E = \frac{\gamma^2 \mu}{2 \ell^2} (\epsilon^2 - 1)$$

$$\gamma = k_e q_1 q_2$$

Coulomb Scattering $- b(\vartheta)$

For hyperbolic orbits

$$r(\varphi) = \frac{C}{1 + \epsilon \cos \varphi}$$

$$\cos \varphi_{\text{max}} = -\frac{1}{\epsilon}$$

$$\varphi_{\text{max}} = \frac{\pi}{2} + \frac{\varphi}{2}$$

$$\cos \left( \frac{\pi}{2} + \frac{\varphi}{2} \right) = -\sin \frac{\varphi}{2}$$

$$= -\frac{1}{\epsilon}$$

Coulomb Scattering $- b(\vartheta)$

Rotate the figure so that $\varphi$ is measured with respect to the x-axis, as in Chapter 10.

$$r(\varphi) = \frac{C}{1 + \epsilon \cos \varphi}$$

As $r \to \infty$, $1 + \epsilon \cos \varphi \to 0$

$$\varphi_{\text{max}} = \frac{\pi}{2} + \frac{\varphi}{2}$$

Coulomb Scattering $- b(\vartheta)$

For hyperbolic orbits

$$r(\varphi) = \frac{C}{1 + \epsilon \cos \varphi}$$

$$\cos \varphi_{\text{max}} = -\frac{1}{\epsilon}$$

$$\varphi_{\text{max}} = \frac{\pi}{2} + \frac{\varphi}{2}$$

$$\cos \left( \frac{\pi}{2} + \frac{\varphi}{2} \right) = -\sin \frac{\varphi}{2}$$

$$= -\frac{1}{\epsilon}$$

Coulomb Scattering $- b(\vartheta)$

Using

$$\sin \frac{\varphi}{2} = \frac{1}{\epsilon}$$

$$E = \frac{\gamma^2 \mu}{2 \ell^2} (\epsilon^2 - 1)$$

$$= \frac{\gamma^2 \mu}{2 \ell^2} \cot^2 \frac{\varphi}{2} \frac{\varphi}{2}$$
Coulomb Scattering \( b(\theta) \)

\[
E = \frac{\gamma^2 \mu}{2 \ell^2} \cot^2 \frac{\theta}{2}
\]

\[
\ell^2 = 2 \mu b^2 E
\]

\[
cot \frac{\theta}{2} = \frac{2(2 \mu b^2 E)E}{\gamma^2 \mu} = \frac{4b^2 E^2}{\gamma^2}
\]

\[
\cot \frac{\theta}{2} = \frac{2bT_0}{\gamma}
\]

Differential Cross Section - Definition

- Consider a small beam striking a large target. The detector has solid angle \( \Delta \Omega \) and detects all particles scattered.

\[
\frac{d\sigma}{d\Omega} = \frac{\text{# of particles detected/sec}}{\text{# of beam particles/sec} \times \text{# target particles} \times \Delta \Omega}
\]

Differential Cross Section - Theoretical

- We usually take 1 target particle...

\[
\frac{d\sigma}{d\Omega} = \frac{\text{# of particles detected/sec}}{\text{# of beam particles} \times \text{area} \times \text{sec}} \times 1 \times \Delta \Omega
\]

Differential Cross Section - Theoretical

1) Find the total number of beam particles per second between \( b \to b + \Delta b \). This is the number in a ring of radius \( b \).

\[
N \equiv \frac{\text{# of beam particles}}{\text{area} \times \text{sec}}
\]

This is: \( N \ 2\pi b \ dB \)

2) Find the total number of scattered particles between \( \theta \to \theta + \Delta \theta \).

\[
b = b(\theta)
\]

\[
\text{dB} = b'(\theta) d\theta
\]

This is: \( N \ 2\pi b \ dB = N \ 2\pi b \ b'(\theta) \ d\theta \)
Differential Cross Section - Theoretical

\[
\frac{d\sigma}{d\Omega} = \frac{\text{# of particles detected/sec}}{N \Delta \Omega}
\]

3) Think of the detector as subtending angles \(\Delta \theta\) and \(\Delta \phi\).
The solid angle is defined by:
\[\Delta \Omega = \sin \theta \Delta \theta \Delta \phi\]
or more generally
\[\Delta \Omega = \int \sin \theta \, d\theta \, d\phi\]

(If the area of the detector is small and a distance \(R\) from the target, this can be expressed in terms of the detector's area: \(A \approx R^2 \Delta \Omega\).)

This is:
\[N \, 2\pi b \, db = N \, 2\pi b' (\theta) \, d\theta\]

Differential Cross Section - Theoretical

The number of particles detected per second is
\[N \, b \, b' (\theta) \, d\theta d\phi\]

\[
\frac{d\sigma}{d\Omega} = \frac{N \, b \, b' (\theta) \, d\theta d\phi}{N \sin \theta \, d\theta d\phi} = \frac{b \, b' (\theta)}{\sin \theta}
\]

Or since we want a positive value, we usually write
\[
\frac{d\sigma}{d\Omega} = \frac{1}{\sin \theta} \left| \frac{db}{d\theta} \right|
\]

Total Cross Section

The total cross section is the differential cross section integrated over all angles.
\[
\sigma = \int \frac{d\sigma}{d\Omega} \, d\Omega = \int \frac{d\sigma}{d\Omega} \sin \theta \, d\theta \, d\phi
\]

Differential Cross Section - Theoretical

4) The number of particles per second scattering into \(d\theta\) is:
\[N \, 2\pi b \, b' (\theta) \, d\theta\]
The number of those hitting the detector is
\[N \, 2\pi b \, b' (\theta) \, d\theta \times \frac{\Delta \phi}{2\pi}\]

Or since we want a positive value, we usually write
\[
\frac{d\sigma}{d\Omega} = \frac{1}{\sin \theta} \left| \frac{db}{d\theta} \right|
\]

This is the usual "book form," but it is usually more convenient to write this as:
\[
\frac{d\sigma}{d\Omega} = \frac{b}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \left| \frac{db}{d\theta} \right|
\]

Hard Sphere Scattering (cm)

\[b = (r_1 + r_2) \sin \frac{\pi - \theta}{2} = (r_1 + r_2) \cos \frac{\theta}{2}\]

\[\frac{db}{d\theta} = -\frac{1}{2} (r_1 + r_2) \cos \frac{\pi - \theta}{2} = -\frac{1}{2} (r_1 + r_2) \sin \frac{\theta}{2}\]

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}} (r_1 + r_2)^2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{1}{4} (r_1 + r_2)^2
\]
**Hard Sphere Scattering (cm)**

\[
\frac{d\sigma}{d\Omega} = \frac{1}{4} (r_1 + r_2)^2 \\
\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \pi (r_1 + r_2)^2
\]

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**Rutherford Scattering (cm)**

\[
b = \frac{\gamma}{2T_0} \cot \frac{\theta}{2}
\]

\[
\frac{db}{d\theta} = -\frac{\gamma}{2T_0^2} \csc^2 \frac{\theta}{2}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}} \left( \frac{\gamma}{2T_0} \right)^2 \cot \frac{\theta}{2} \csc^2 \frac{\theta}{2}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{1}{\sin^2 \frac{\theta}{2}} \left( \frac{\gamma}{4T_0} \right)^2 = \frac{\gamma^2}{16T_0^2 \sin^4 \frac{\theta}{2}}
\]

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**CM to Lab Conversion**

- The only thing that changes in lab vs cm is \( \Delta \Omega \).

\[
\frac{d\sigma}{d\Omega} = \frac{\# \text{ of particles detected}/sec}{\# \text{ of beam particles}/sec \times \# \text{ target particles}/area} \times \Delta \Omega
\]

\[
= -\Delta (\cos \delta_{lab}) \Delta \phi_{lab} \\
\Delta \Omega_{cm} = -\Delta (\cos \delta_{cm}) \Delta \phi_{cm} \\
\Delta \phi_{lab} = \Delta \phi_{cm} \\
\rightarrow \frac{d\sigma}{d\Omega_{lab}} = \frac{d\sigma}{d\Omega_{cm}} \times \frac{d(\cos \delta_{cm})}{d(\cos \delta_{lab})}
\]