Physics 321 – Sample Midterm Test #2

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Name: _________________________________

· You may use a calculator as long as you do not used stored information in it.
· I will not grade anything written on the colored cover sheet from the Testing Center.
· Be sure your copy of the test has 7 problems.
· Please answer each question completely; however, there is no need to be wordy.
· If you ever get bogged down in the math, explain in detailed words what you are trying to do.

Possibly useful information:

\[ \vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}' \]
\[ \vec{v}' = \vec{v} - \vec{\omega} \times \vec{r} \]
\[ \vec{a} = \vec{a}' + \vec{\omega} \times \vec{r}' + 2\vec{\omega}' \times \vec{r}' + \vec{\omega}' \times (\vec{\omega} \times \vec{r}') \]
\[ \vec{a}' = \vec{a} - \vec{\omega} \times \vec{r} - 2\vec{\omega} \times \vec{r}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \]
\[ m\ddot{r}' = \vec{F}_{real} + m\vec{\omega}' \times \vec{r}' + 2m\dot{\vec{r}}' \times \vec{\omega}' + m(\vec{\omega}' \times \vec{r}') \times \vec{\omega}' \]

Small angle approximation: \( \sin \theta \approx \theta \) \quad \cos \theta \approx 1

Do your work on the blank pages. Only put the major steps (not all the algebra, simplifications, steps in evaluating integrals, etc.) on the “front” pages.
1. (15 points) For each of the following illustrations, write down the Lagrangian and the equation or equations of motion based on the Lagrangian. Create your own symbols for necessary parameters (masses, radii, angles, spring constants, etc.). You do not need to evaluate any moments of inertia. You can choose your favorite coordinates. Sketch the coordinates and other important information on the figure so I can more easily interpret your solution. Ignore frictional and drag forces. Be sure your equations of motion could be solved to give a valid answer.

(a) Ball on an incline. The incline doesn’t move and the ball rolls without slipping.

Lagrangian:
\[ \mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{I}{R^2} \dot{\theta}^2 - mg x \sin \theta \]

Equation(s) of motion:
\[ \ddot{x} = -\frac{mg \sin \theta}{m+\frac{I}{R^2}} \]
\[ \dot{\theta} = -\frac{mg \sin \theta}{I/R^2} \]

(b) Mass on a vertical spring. The spring is massless. (The wood represents the ceiling from which the block is suspended.)

Lagrangian:
\[ \mathcal{L} = \frac{1}{2} m \dot{y}^2 + m g y - \frac{1}{2} k (y - \ell_0)^2 \quad (y = 0 \text{ is unstretched}) \]

Equation(s) of motion:
\[ m \ddot{y} = m g - k y \]

(c) Coupled masses on springs. The wood does not move.

Lagrangian:
\[ \mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 (\dot{x}_1 + \dot{x}_2)^2 - \frac{1}{2} k (x_1 - \ell_1)^2 - \frac{1}{2} k (x_2 - \ell_2)^2 \]

Equations(s) of motion:
\[ m_1 \ddot{x}_1 + m_3 (\ddot{x}_1 + \ddot{x}_2) = -k (x_1 - \ell_1) \]
\[ m_2 \ddot{x}_2 + m_3 (\ddot{x}_1 + \ddot{x}_2) = -k (x_2 - \ell_2) \]
\[
(1) \quad y = x \sin \theta \\
T = \frac{1}{2} m x^2 \quad U = mgy = mgx \sin \theta
\]
2. (20 points) A massless string is coiled around a pulley so that the block attached to it causes the pulley to rotate as it unwinds. Let the mass of the block be $m$, the moment of inertia of the pulley be $I$, and the radius of the pulley be $R$. The position of the pulley is $y$ which is positive downward. Call the rotation angle of the pulley $\theta$. (The bracket holding the pulley to the ceiling is drawn.)

The Lagrangian for this system is:

$$\mathcal{L} = \frac{1}{2}\mu y^2 + mg y$$

(a) What is $\mu$ in terms of the quantities given above?

$$\mu = m + \frac{I}{R^2}$$

(b) What is the momentum conjugate to $y$?

$$p = \frac{\partial\mathcal{L}}{\partial \dot{y}} = \mu \dot{y}$$

(c) Write the Hamiltonian for the system.

$$H = \frac{p^2}{2\mu} - mg y$$

(d) Find Hamilton’s equations of motion.

$$\frac{\partial H}{\partial y} = -\dot{\dot{y}} = -mg \quad \dot{\dot{y}} = mg$$

$$\frac{\partial H}{\partial p} = \dot{y} = \frac{p}{\mu}$$
\[ T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2} I \dot{\varphi}^2 = \frac{1}{2} \left( m + \frac{I}{\ell^2} \right) y^2 \]
3. (20 points) Find the tension in the string of Problem 2 using Lagrange multipliers.

Equation of constraint:

\[ \begin{align*}
\frac{\partial L}{\partial \phi} &= \lambda \\
-\rho \frac{d^2 \phi}{dt^2} &= \lambda \\
\frac{\partial L}{\partial \dot{\phi}} &= \lambda
\end{align*} \]

Lagrangian that is used with Lagrange multipliers:

\[ L = \frac{1}{2} m \ddot{\phi}^2 + \frac{1}{2} I \dot{\phi}^2 + m g \phi \]

Equations of motion including Lagrange multiplier term(s):

\[ \begin{align*}
\frac{d^2 \phi}{dt^2} &= \frac{\dot{\phi}}{R} \\
\frac{d^2 \phi}{dt^2} &= \frac{\dot{\phi}}{R} \\
(m + I) \ddot{\phi} &= m g \\
\mu \ddot{\phi} &= m g
\end{align*} \]

Tension in the string:

\[ T_{\text{string}} = \lambda = \frac{m \ddot{\phi}}{\mu} - m g \]

\[ = m \left( \frac{m g}{\mu} - m g \right) = m g \left( \frac{m - \mu}{\mu} \right) - \frac{m g (m - \mu)}{\mu R^2} \]

\[ = -\frac{m g I}{\mu R^2} \]
4. (10 points) The Lagrangian for this system (in a small angle approximation) is:

\[ \mathcal{L} = \frac{1}{2} m \ell^2 \dot{\theta}_2^2 + \frac{1}{2} m \ell^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 - \frac{1}{2} m g \ell \theta_1^2 - m g \ell \theta_2^2 \]

(a) What are the momenta conjugate to each variable?

\[ p_1 = m \ell^2 (\dot{\theta}_2 + \dot{\theta}_1) \]
\[ p_2 = m \ell^2 \dot{\theta}_2 + m \ell^2 (\dot{\theta}_1 + \dot{\theta}_2) \]

(b) What is the Hamiltonian? (Note: You don’t need any hard algebra.)

\[ H = \frac{(p_2 - p_1)^2}{2 m \ell^2} + \frac{p_1^2}{2 m \ell^2} + \frac{1}{2} m g \ell \theta_1^2 + m g \ell \theta_2^2 \]

\[ p_2 - p_1 = m \ell^2 \dot{\theta}_2 \]
5. (20 points) In this problem you need to find the Lagrangian for a more difficult configuration. A mass $M$ has a hole drilled through it so it can slide (frictionlessly) over a horizontal rod. The mass is connected by a spring to the wall. The coordinate $X = 0$ when the spring is unstretched.

A pendulum is attached to the bottom of the mass. The length of the pendulum is $L$. It makes an angle $\theta$ with respect to the vertical. Note that $0^\circ$ is downward. The pendulum rod is massless and there is a point mass $m$ at the end of the rod.

Big hint: This problem is hard unless you think of the $x$ and $y$ coordinates of $m$ when you consider the pendulum’s kinetic energy.

(a) Write the kinetic energy in terms of $X$ and $\theta$.

$$T = \frac{1}{2} k \dot{X}^2 + \frac{1}{2} m \dot{X}^2 + \frac{1}{2} m L^2 \dot{\theta}^2 + m \dot{X} \dot{\theta} L \cos \theta$$

(b) Write the potential energy in terms of $X$ and $\theta$.

$$U = \frac{1}{2} k X^2 - m g L \cos \theta$$

(c) Using the Lagrangian, find the equations of motion.

$$\begin{align*}
(M + m) \ddot{X} + m \ddot{\theta} L \sin \theta - m \dot{\theta} \dot{L} \sin \theta &= -k \ddot{X} \\
ml^2 \ddot{\theta} + mL \ddot{\theta} L \sin \theta - mL \dot{\theta} \dot{L} \sin \theta &= -m \dot{L} \ddot{\theta} L \sin \theta - mL \dot{\theta} \dot{\theta}
\end{align*}$$
\[ x = \sqrt{x^2 + y^2} \quad y = -L \cos \theta \]
\[ x = \sqrt{x^2 + y^2} \quad y = L \sin \theta \]
\[ x^2 + y^2 = x^2 + 2x\dot{x} \cos \theta + L^2 \theta^2 \]
6. (10 points) A bead on a turntable slides without friction along a thin rod that is directed radially outward. A spring attached to the bead is unstretched when the bead is located at \( r' = a \) where \( r' \) is the distance of the bead from the center of the turntable. The spring constant is \( k \), the mass of the bead is \( m \), and the turntable rotates at an angular speed \( \omega \) in a clockwise direction.

(a) Find an equation of motion for \( r'(t) \). (Note: This is a scalar and your equation should have no vectors remaining in it!)

Hint:
\[
\ddot{r}' = \ddot{a}' + \dot{\omega}' \times r' + 2\dot{\omega}' \times \dot{r}' + \ddot{\omega}' \times (\dot{\omega}' \times r')
\]
\[
\ddot{a}' = \ddot{a} - \dot{\omega} \times \dot{r} - 2\dot{\omega} \times \ddot{r} + \dddot{\omega} \times (\dot{\omega} \times \ddot{r})
\]
\[
\dot{r}' = -k / m \ (r' - a) + \omega'^2 r'
\]

(b) Is there a Coriolis force on the bead? If not, explain why not. If so, describe its direction and its effect.

When the bead moves in, the force is tangential and clockwise. If it moves out, it reverses. In this case, all it does is push against the rod so the rod has to exert a normal force sideways on the bead.

7. (15 points) A general equation for the forces in a rotating coordinate frame is given below:

\[
m\ddot{r} = \vec{F} \quad \text{real force}
\]
\[
+ m\dot{r} \times \dot{\Omega} \quad \text{tangential force}
\]
\[
+ 2m\dot{r} \times \dot{\Omega} \quad \text{Coriolis force}
\]
\[
+ m(\ddot{\Omega} \times \dot{r}) \times \ddot{\Omega} \quad \text{centrifugal force}
\]

Fill in the blanks with the name for each term.

Assume you are sitting on a turntable facing inward toward the axis of rotation. The turntable is rotating to your right (counterclockwise when viewed from above). What is the direction of the angular velocity \( \dot{\Omega} \) ?

**upward**

Assume you are turning faster and faster and at the same time you lean forward in your seat (so in the reference frame of the room, your velocity is toward the center of the room). What is the direction of each term in this equation?

\[
m\ddot{r} = \vec{F} \quad z: \text{gravity and normal forces - zero net, } r: \text{normal and friction =centripetal}
\]
\[
+ m\dot{r} \times \dot{\Omega} \quad \text{left}
\]
\[
+ 2m\dot{r} \times \dot{\Omega} \quad \text{right}
\]
\[
+ m(\ddot{\Omega} \times \dot{r}) \times \ddot{\Omega} \quad \text{back}
\]