Physics 321
Sample Final Exam

Name: _________________________________

· You may use a calculator as long as you do not used stored information in it.
· Be sure your copy of the test has seven problems.
· Please answer each question completely; however, there is no need to be wordy.
· If you ever get bogged down in the math, explain in detailed words what you are trying to do.

Possibly useful information:

\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

For small angles: \( \sin \theta \approx \theta \), \( \cos \theta \approx 1 - \frac{1}{2} \theta^2 \).

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}} \left| \frac{db}{d\theta} \right|
\]

\[
\Gamma_1 = l_{11}\dot{\omega}_1 - (l_{22} - l_{33})\omega_2\omega_3
\]

\[
\Gamma_2 = l_{22}\dot{\omega}_2 - (l_{33} - l_{11})\omega_3\omega_1
\]

\[
\Gamma_3 = l_{33}\dot{\omega}_3 - (l_{11} - l_{22})\omega_1\omega_2
\]

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Do your work on the blank pages after each problem. Only put the major steps (not all the algebra, simplifications, steps in evaluating integrals, etc.) on the “front” pages.
1. Find the Lagrangian for each of the systems below. Strings are massless; there is no sliding friction and no slipping. Make up your own coordinates and variable names if they’re not given, but label them on the figures. Do not use a small angle approximation. Do not find equations of motion – only the Lagrangian.
2. The moment of inertia of a lamina is given by the formula
\[ I = \int d^2 \sigma dA \]
where \( \sigma \) is the mass per unit area of the lamina. (Recall that a lamina is a thin, flat sheet of material.)

(a) A square lamina of length \( a \) on a side and mass \( m \) is suspended by one corner to make a physical pendulum. What is its moment of inertia about the point of suspension? Your answer should be in terms of \( m \) and \( a \).

(b) Find the inertia tensor for this lamina with respect to the axes indicated.

(c) Find the principal axes. I want to see your work, so don’t just write down an educated guess.
Blank page 2.
3. Write down the three equations for the motion of a baseball in midair in three dimensions. Include linear and quadratic drag forces. Take the vertical direction to be the $z$ direction and assume a wind is blowing in the $y$ direction. (Use $c_1$ and $c_2$ for the linear and quadratic drag coefficients. These are not functions of the ball diameter.)

4. (a) Newton’s Laws are fully deterministic. That is, if you solve the same differential equation with the same boundary conditions, you get the same answer every time. So what does it mean that a system is chaotic in classical mechanics? List the two major points we discussed in class.

(b) As a damped, driven oscillator approaches chaos, something strange starts happening to its phase (state) space plots. What is that?

(c) When we plot a phase space plot in terms of $\sqrt{T}$ vs $\sqrt{U}$, the plot spirals inward. What does that tell you about the system?

(d) When we solve a non-linear equation by the method of successive approximations, we find that solutions are no longer limited to a single frequency. What important characteristic do the solutions have?
5. A double pendulum consists of two masses $m$ each attached to a rod of length $d$. In the small angle approximation, the Lagrangian can be written as
\[ L = \frac{1}{2} m d^2 \dot{\theta}_1^2 + \frac{1}{2} m d^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 - m g d \left( \theta_1^2 + \frac{1}{2} \theta_2^2 \right) \]

(a) Show that the equations of motion can be reduced to:
\[
2\ddot{\theta}_1 + \ddot{\theta}_2 = -2\omega_0^2 \theta_1 \\
\ddot{\theta}_1 + \ddot{\theta}_2 = -\omega_0^2 \theta_2
\]

(b) Describe how you would use Mathematica to find the frequencies and normal modes of the system.

(c) By hand, find the frequencies of the normal modes in terms of $\omega_0$.

(d) The eigenvectors of the system are $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. What does that tell you? Which eigenvector has the higher frequency?
6. (a) Using Euler’s equations, find the equations of motion for a football with moments of inertia $I_{33}$ about an axis passing through the tips of the football and $I_{11}$ about an axis perpendicular to this axis and passing through the football’s center of mass. Express your answer as a series of differential equations in terms of the moments of inertia and the components of angular velocity ($\omega_1, \omega_2, \omega_3$) and their derivatives.

(b) You toss a football that wobbles along its trajectory. What can you say about the direction of the angular momentum vector (as measured with respect to field coordinates)? What about the angular velocity vector and the direction of the football’s symmetry axis? (Ignore things like the center of mass motion and air drag.)

(c) What terms do we use to describe the angular velocities, $\phi, \theta, \psi$, respectively?
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7. An incident particle of mass \( m \) and velocity \( v_0 \) in the \( z \) direction collides with a target particle of mass \( M \) and scatters elastically. The target particle is initially at rest.

(a) Find the velocity of the center of mass, \( v \). Express your answer in terms of \( m \), \( M \), and \( v_0 \).

(b) The impact parameter is related to the center of mass scattering angle by the relation
\[
b = a \cos \frac{\theta}{2}
\]
where \( a \) is a constant. Find the differential cross section in the center of mass frame.