Lesson 2 – Moving Point Charges and Their Forces

2.0 Introduction

It might not be too surprising that motion affects the force between point charges. Motion usually changes forces only a small amount; however, these effects cannot be ignored when either 1) particles are moving at a large fraction of the speed of light, or 2) a small force from billions of moving particles adds up to make something significant. We usually encounter fast-moving charges only in particle accelerators or cosmic rays. On the other hand, the force between permanent magnets, for example, is caused by the motion of large numbers of slowly moving particles.

In spite of the motion of particles, we will find that the Thread Model can be used both qualitatively and quantitatively to describe the force between point charges. If a field particle is at rest, then all we need to know is how to find the length and density of threads emitted by moving source particles. However, if the field particle is moving, some complications arise that require the use of the special theory of relativity. We'll leave many of the proofs to an appendix, but we will learn a few things about relativity to help us understand why these forces behave the way they do. To do this, we will introduce a new component of the Thread Model that makes it easy to account for the motion of the field particles. This new component is something called a "stub," a line segment attached to the head of each thread. The motion of the source determines the stub, and the motion of the field particle combined with the stub gives an additional force we need to consider. The thread force is the electric force and threads form electric field lines. The stub force is the magnetic force and stubs form magnetic field lines.

2.1 How Motion Modifies Threads

In the last chapter, we learned how threads were emitted from source particles at rest. Now we need to visualize the same process for a moving source particle. We let the source particle and the field particle both be positively charged so the force is repulsive. For simplicity, we also assume that the source particle is moving along the x axis. At time $t = 0$ the source is at $x = 0$, the origin of a coordinate system. We consider a thread emitted at this time. (See Fig. 2.1.) Let's call the position of the source when it emits the thread S. The head of the thread is emitted at an angle θ with respect to the direction the source travels. A short time later, the source has moved to a new position T where it emits the tail of the thread. The tail also is emitted at the angle θ. Somewhat later, at time t, the source moves to point U and the head moves to point P. The path followed by the head is called the "head line," written as $\vec{r}_h$. The path followed by the tail is called the "tail line," $\vec{r}_t$. Note that the head line and the tail line are parallel to each other and that the thread remains constant in length and direction. (The location of the thread at different times is shown by the light orange arrows.) The vector length of the thread is written as $\vec{\ell}$.

All of that probably seems a little confusing, but carefully look at Fig. 2.1 to get a feeling for the geometry of the system. Remember that in the time interval 0 to t, the head moves from S to P while the source moves from S to U. Also note the angle θ is the angle between the direction of motion (the x axis in this case) and the head line. The angle ψ is defined as the angle between the direction of motion and the ray line.

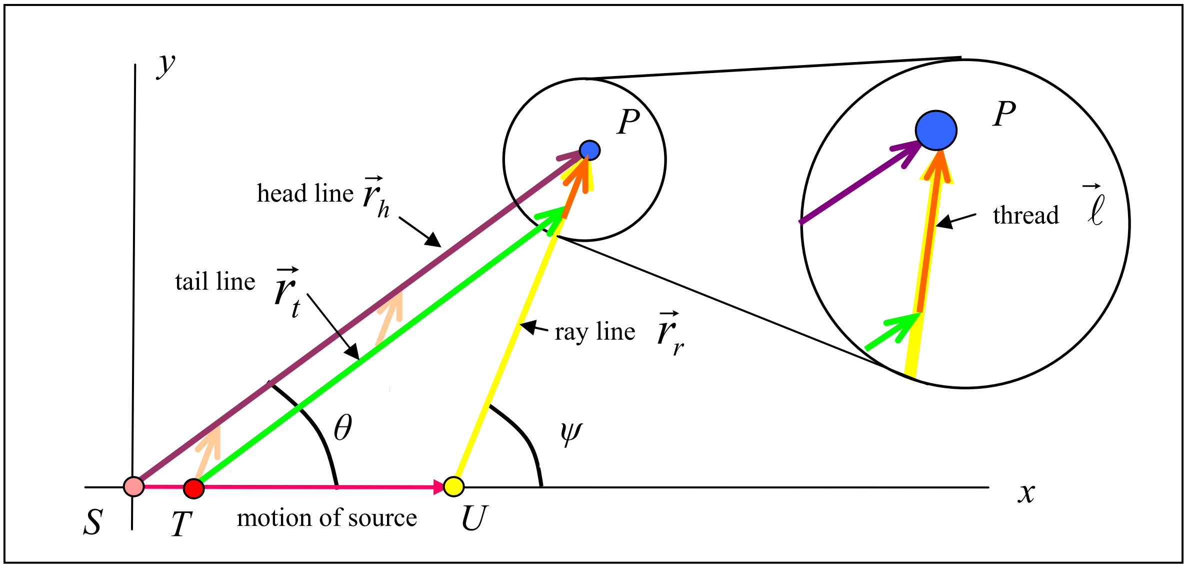

Figure 2.1. A thread emitted by a source particle moving in the $+\hat{x}$ direction.

Since threads are not stationary objects, it's helpful to look at an animated version of Fig.2.1. This can be found at:
http://www.physics.byu.edu/faculty/rees/220/AVIfiles/OneThread.avi

We see that the length of the thread is affected by the motion of the source, but in a rather complicated way. We'll get back to that a little later. However, the direction of the thread is easily determined, as the thread must lie along the line that joins point P with point U. (Recall that P is the position of the field particle at time t and that U is the position of the source particle at the same time.) That means that if we "take a picture" of the particles and threads at time t, the direction of the thread – which is also the direction of the force – is still radially outward from the source particle. However, we do need to be careful to specify that the direction is radially outward from where the source is located at time t **not** radially outward from where the source was at time 0 when the thread was first emitted! If you think of the threads like little balls being thrown outward from the source, then you would expect the force to be along the head line; however, this is **not** the case. The direction of the force is along the **ray line**.

Photograph of laser passing through chalk dust.

Figure 2.2. Laser beam passing through chalk dust.

If I explain why the ray line is called the ray line, it might help you visualize the threads. When we shine a laser through some dusty air (in Fig. 2.2, we just hit some chalk erasers together), we can see a ray of light emanating from the laser. Of course, the reason for this is that the light reflects off from the small dust particles in the air. If the laser is at rest, then each photon travels along the same path. This path is what we see as the ray of light in the figure. Now imagine the laser is traveling at half the speed of light while we shine a beam off at an angle θ, as shown in Fig. 2.3. Due to the motion of the source, each photon travels along a different path. A few of these paths are depicted by the dashed lines in the figure. But what we would see in a photograph as the ray of light is the line that joins the all the illuminated dust particles together. This is the solid line in the figure. This line eventually comes back to the laser itself, since the last photons emitted are right next to the laser at the instant the picture is taken. Let's recapitulate all this: The photons emitted at S_1 arrive at P_1 when the picture is taken, photons emitted at S_2 arrive at $P_{2,}$ and photons emitted at S_3 arrive at $P_{3.}$ The photons move along the dashed lines, but the ray that shows in the photograph is the solid, red line. Threads are like small sections of the laser beam, and hence they lie along the ray line rather than along the head line.

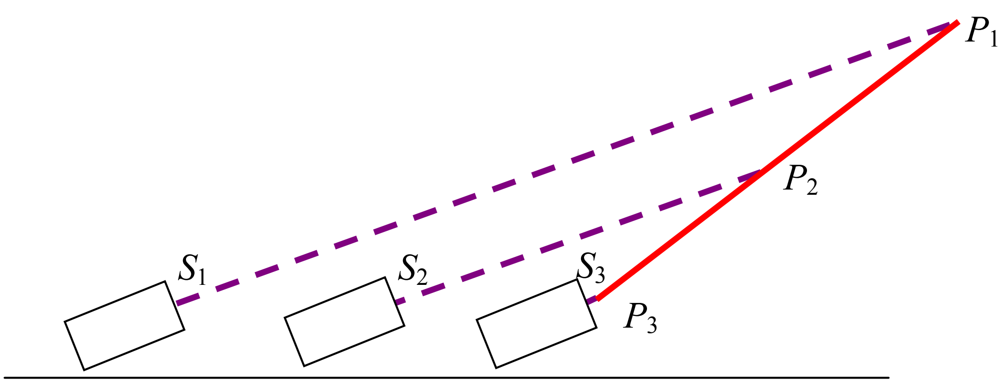

Figure 2.3. Photon paths and light ray of a laser traveling half the speed of light.

To compare the differences between a moving source and a stationary source, let's look at what happens when the source is at rest. (See Fig. 2.4.)

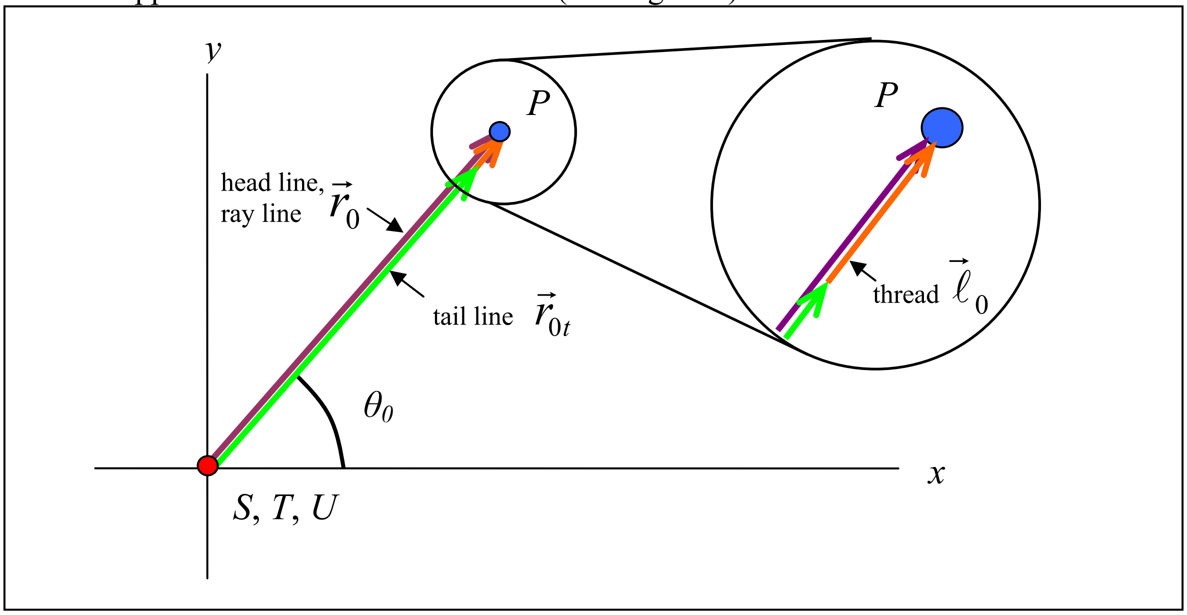

Figure 2.4. A thread emitted by a source particle at rest.

In this case, the points *S*, *T*, and *U* are in the same place. The head line and the ray line are identical and the only difference between the head line and the tail line is that the tail line is a little shorter. The head line and tail line both make an angle θ_0 with respect to the *x* axis. We'll make extensive use of Figs. 2.1 and 2.4 a little later, but for now the most important thing to notice is that the ray line and the head line are in the same direction when the source is at rest, but in rather different directions when the source is moving.

ength and
d rce. In Fig.
2 f light.

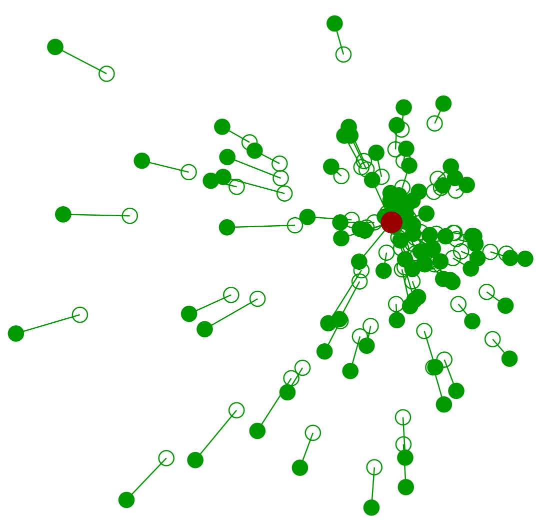

Figure 2.5. The threads of a source particle moving at 70% of the speed of light.

To get a better understanding of these threads, look at the following two animations of sources traveling at 20% of the speed of light and 70% of the speed of light.

Uncompressed versions:

v=0.2c http://www.physics.byu.edu/faculty/rees/220/AVIfiles/Threads2.avi
v=0.7c http://www.physics.byu.edu/faculty/rees/220/AVIfiles/Threads7.avi

Note that all the threads point back toward the source charge. Note also that the heads are quite concentrated in front of the source and rather sparse behind it. On the other hand, threads in front of the source are quite short while those behind the source are fairly long. Since the force resulting from the threads is proportional to the thread density multiplied by the thread length, the two contributions have opposite effects.

This is borne out Fig. 2.6 which shows the force on a test particle located a fixed distance from the source but at various angles ψ (see Fig. 2.4). The different curves are from source particles traveling at different speeds. The speeds are expressed in terms of the parameter β which is defined by the relation

(2.1 Definition of β)
$$\beta \equiv \frac{v}{c}$$

where c is the speed of light. Thus a particle traveling at 70% of the speed of light has β = 0.7. As the source travels faster, the force becomes larger at 90°and smaller at 0° and 180°. This is because relativity causes the threads to "clump up" near 90°. So as the source goes faster, the forces on field particles in front and behind the source particle are rather small compared to the force on field particles above, below, and to the sides of the source particle.

Note also that the curves are symmetric about 90°. This means that forces are the same on field particles located at the same distance from the source, but at angles of ψ = 20º and ψ = 180º – 20º =160º, for example. As we noted above, the density of threads is large at 20º and the length of the threads is large at 160º. These effects just balance out to make the magnitude of the forces identical at these two angles.

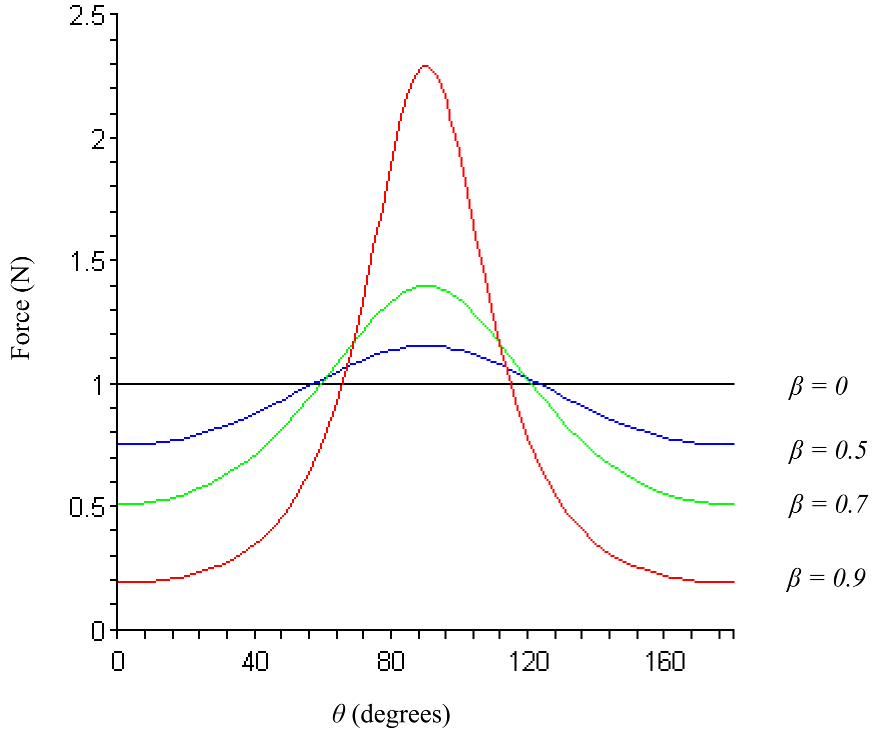

Figure 2.6. The force at a fixed distance from a moving test particle as a function of ψ, given for several values of β.

Things to remember:

- Draw a picture of a thread emitted by a moving source particle. Identify the head line, the tail line, and the ray line, and the angles θ and ψ.
- Threads of a moving particle lie along the ray line.
- Threads are shorter in front of a moving particle and longer behind it.
- Threads are more dense in front of a moving particle and less dense behind it.
- These effects balance, making the force on a field particle the same at forward and backward angles ($\psi = 20^o$ and $\psi = 160^o$, for example).
- Threads clump up near $\psi = 90^o$, so the force is larger there.

2.2 Threads and Moving Train Cars

Hopefully by now you are beginning to visualize the threads emitted by moving sources. But just to belabor the point a little, we're going to think of charged particles emitting threads in train cars. Our "basic train car" will be moving in the $+x$ direction. The length of the car (in the x direction) is L and the height of the car is H as shown in Fig. 2.7. We place a charged particle at point S in the center of the lower left edge of the car. Opposite the charge, in the center of the upper right edge of the car is a point P. The distance between S and P is r_0. At time $t = 0$, a thread is emitted from point S. (We'll think of the thread as essentially a point on the scale of the room, so we won't bother drawing the thread's tail, etc.) The angle with which the thread is emitted is θ_0, as in Fig. 2.4.

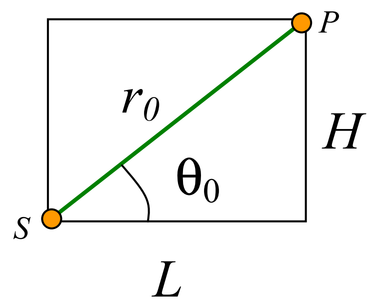

Figure 2.7. Geometry of the train car in the rest frame.
A thread follows the green line from S to P.

Now let's look as this train car from the viewpoint of an observer who sees the car moving to the right with a speed v. *We will not use relativity in this section, so the results will need to be modified; however, I want you to understand the problem before we add the complications of relativity.*

The moving observer sees the thread leave point S at time $t = 0$, and she sees the thread take the green path from S to P as shown in Fig. 2.8. This path has a length r_h. This is the head line. The angle the head line makes with respect to the x direction is θ. When the thread arrives at point P, the source – at the left edge of the car – has moved to a point U. The line from U to P is the ray line. The ray line still has a length of r_0. Furthermore, the ray line makes an angle ψ with respect to the x direction. Since the dimensions of the train car have not changed at all, ψ and θ_0 are identical. As we can see from the diagram, $\theta < \psi = \theta_0$.

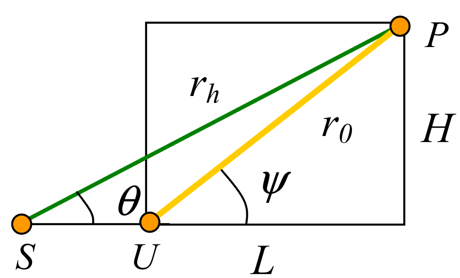

Figure 2.8. Geometry of the train car viewed by an observer
who sees it moving to the right with velocity v.

You can view an animation of this process online at
http://www.physics.byu.edu/faculty/rees/220/graphics/boxthrnr.gif.
In this animation, there are two train cars with $H = L$ so that threads are emitted at a rest angle of θ_0 =45° in each case. One train car is at rest while the other train car moves at 60% of the speed of light. Note that all the clocks read the same time. The black lines are the ray lines for each train car, and the blue line is the head line for the moving car. Note that the thread emitted by the moving source travels a larger distance than the thread in the stationary train car, but both arrive at point P at the same time. That tells us that the thread emitted by the moving source travels at a speed greater than the speed of light (from a stationary source). But relativity tells us that it is impossible to travel faster than light, so something is wrong with this picture!

Things to remember:
• Be sure you understand the head line and the ray line for both stationary and moving cars. We'll redo the math using relativity!

2.3 A Little Relativity

At the beginning of the 20th Century, almost everything seemed to be neatly explained by the physical theories that had been developed in the past several years. However, physicists were puzzled over a few phenomena that didn't behave as expected. One such problem was that electromagnetic theory didn't always agree with Newtonian physics. In 1905, Albert Einstein was concerned with precisely the question that we are discussing in this chapter, the forces between moving charged particles. In his paper entitled *On the Electrodynamics of Moving Charges*, Einstein introduced a new theory to explain the inconsistencies, the theory of relativity. To distinguish it from general relativity, his later work with gravitation and curved space, we now call the original theory based on flat space the "special theory of relativity" or just "special relativity."

As it turns out, special relativity was already built into electromagnetic theory, so the discrepancies were the result of flaws in Newtonian mechanics. because of that, you really need to use relativistic mechanics to describe forces between charged particles. Depending on your background in physics; however, you may or may not have studied much about special relativity. I won't explain relativity in detail, but it will be important to know a few basic facts.

When Einstein first derived special relativity, he began with two postulates. these were:

1) The Principle of Relativity (also called Galilean Relativity): the mathematical form of physical laws is the same for any observer who is not undergoing acceleration. Einstein's understanding of this was that there is no such thing as an object, a platform, or even an "ether" throughout space that could be considered to be absolutely at rest. As long as you're not accelerating, you are just as much at rest as anyone can be. There is no possible experiment that could tell you that you are at rest or moving with constant speed. This basic idea had been accepted in some form since Newton's time.
2) The constant velocity of light: If you are on a train going 40 m/s and throw a ball in the direction of the train's motion at 20 m/s relative to the train, an observer on the ground would measure the velocity of the ball to be 60 m/s. However, if you are on a spaceship going half the speed of light (1.5×10^8 m/s) with respect to a platform in space and shine a light in the direction of motion, an observer on the platform would measure the speed of the light to be 3×10^8 m/s, not 4.5×10^8 m/s. This was an experimental result and had to be taken as a postulate, but it still made many people uneasy in the beginning.

Relativity is closely tied with the question of how observers moving at different velocities measure the same physical quantities. Before we continue, it's helpful to define a few terms.

1) *a reference frame* is a three-dimensional Cartesian coordinate system with a synchronized clock available at every point in space.
2) *an inertial reference frame* (or just *an inertial* frame) is a reference frame that is not accelerating, so the standard laws of physics must apply in it.

3) *a rest frame* is an inertial reference frame in which an object is at rest.
4) *space-time* is the four-dimensional space that includes the three dimensions of normal (configuration) space and the one dimension of time.
5) β (*beta*) is the ratio of a velocity to the speed of light, $\beta = \frac{v}{c}$, as we saw above.
6) γ (*gamma*) is a particular function of β that appears so regularly in equations it was given a special name. $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. When $\beta = 0$, then $\gamma = 1$. γ remains very close to 1 until β is greater than about 0.1. (Hence, relativistic effects are usually small for objects traveling less than about 10% of the speed of light.) As β approaches 1, γ becomes infinite.

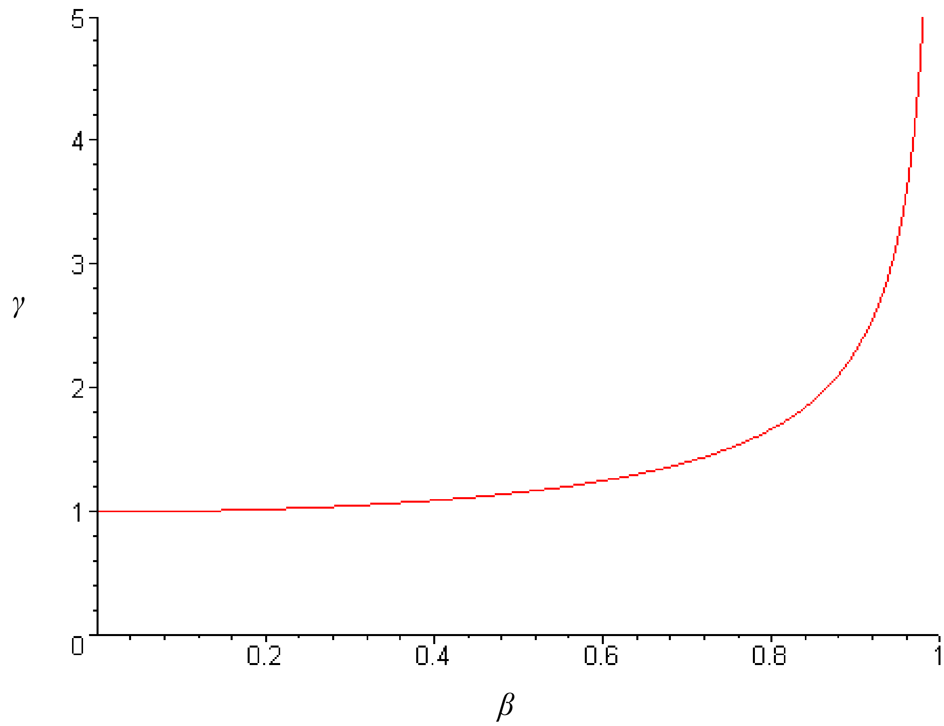

Figure 2.9. γ as a function of β.

Some important results of relativity are:
1) *length contraction*. Moving rods are shorter. If you see a rod moving with a velocity β, its length is $\ell = \frac{\ell_0}{\gamma}$ where ℓ_0 is its length when it is at rest. If an object is moving in the x direction, length contraction only affects the x dimension of the object. The other dimensions are not contracted.
2) *time dilation*. Moving clocks run slowly. An interval of time for an object moving with velocity β is measured to be $\Delta t = \gamma \Delta t_0$ where Δt_0 is the length of the same interval in the

object's rest frame. Thus, if an elementary particle decays with a short half-life when is at rest, it decays with a longer half-life when it is moving.

3) *mass increase*. Moving objects become more massive. The mass of an object moving with velocity β is measured to be $m = \gamma m_0$ where m_0 is its rest mass.
4) *energy-mass* equivalence. This one you know: $E = mc^2$. Note that *m* is the moving mass, not the rest mass.
5) *constant c*. The speed of light in vacuum is the same for any observer, regardless of the motion of the source or the observer.
6) *maximum speed*. No object or signal can travel faster than the speed of light. Massive objects can never quite reach the speed of light, as their mass becomes infinite, their length zero, and their clocks stop.
7) *relativity of simultaneity*. That's a useful expression if you want to impress someone! This means that two events that are simultaneous in one inertial frame are not necessarily simultaneous in another reference frame. (This one causes students a great deal of headache, as it is very easy to assume simultaneity when you shouldn't.)

These are just some very basic ideas from relativity. We'll apply a few of them conceptually in this course, but you won't have to work any difficult relativity problems. If I ask you what length contraction means or if time dilation is one consequence of relativity, however, I would expect you to be able to respond correctly.

Things to remember:

- $\beta = v/c$. $\gamma = 1/\sqrt{1-\beta^2}$. Note that $\beta \le 1, \quad \gamma \ge 1$.
- As an object approaches the speed of light, its length shortens, its mass increases, and it clocks slow down, all by a factor of γ.
- The speed of light is a constant with respect to any observer. No object or signal can travel faster than the speed of light.
- Be able to describe a reference frame and an inertial frame. The laws of physics are the same in all inertial frames.
- Two events that are simultaneous in one inertial frame are not necessarily simultaneous in another.

2.4 The Relativistic Train Car

In Section 2.2, we concluded that when a thread travels from one corner of a moving train car to the other, the velocity of thread measured by an observer on the ground must be greater than the speed of light. Since nothing can travel faster than the speed of light, we know that there must be some flaws with the conclusions we reached previously. In this section, we will see how relativity changes our former conclusions.

The main difference in the two cases is that the train car is contracted in the direction of motion by a factor of γ.

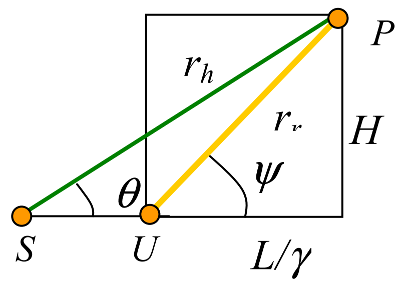

Figure 2.10. Relativistic geometry of the train car in a frame where it is moving to the right with velocity β.

Since the car is contracted, the angle ψ is nearer 90˚ than it is in the nonrelativistic case. In general, we see that:

$$\theta \le \theta_0 \le \psi \ .$$

We also have an animation of this process. This animation can be found at http://www.physics.byu.edu/faculty/rees/220/graphics/boxthread.gif. The clocks in the rest frame all read the same values so only one clock is shown. However, the clocks on opposite sides of the moving car read differently, a result of the relativity of simultaneity. When the thread emitted in the stationary car arrives at the opposite corner, the car's clock reads 6 o'clock. When the thread emitted in the moving car reaches the opposite corner, the clock in the moving car also reads 6 o'clock. The clock in the stationary car reads about 10 o'clock but the distance traveled by the thread is longer. In each of these cases, the speed of light is measured to be exactly the same!

Things to remember:
• Relativity makes the angle ψ closer to vertical (90°) than is θ_0.

2.5 Moving Source and Stationary Field Particle

When the source particle is moving but the field particle is stationary, we can find the force on the particle by using the same technique as in Lesson 1. The only difference is that we now have to modify the formulas for the length and the density of the threads. The force on a field particle at rest caused by a moving source particle is:

(2.2)
$$\vec{F}_{thread} = \frac{e}{\varepsilon_0} q_f \vec{\ell} \nu$$

$\vec{F}_{thread}$ is the (vector) force on the field particle in newtons (*N*).
e is the charge of an electron, 1.602×10^{-19} *C*.
ε_0 is 8.85×10^{-12} C^2/Nm^2.
q_f is the charge of the field particle in coulombs (*C*).
$\vec{\ell}$ is the (vector) length of a thread at the field particle. It has units of meters (*m*)
ν is the density of threads at the field particle in units of $1/m^3$.

To determine the length of a thread, let's consider the emission of a thread in the frame where the source is moving in the *x* direction. Fig. 2.12 (a) shows the thread just at the instant the tail is emitted. The distance from *S* to *Q* is the speed of light times the time it takes for the thread to be emitted. In the rest frame of the source, the thread's head moves a distance ℓ_0, the thread length, by the time the tail is emitted. Since the thread travels at the speed of light *c*, the time it takes to emit the thread in the rest frame is:

$$\Delta t_0 = \frac{\ell_0}{c}.$$

To convert this to the time in the frame where the source is in motion (the lab frame), we need to remember that moving clocks run slow by a factor γ. That means the time measured in the lab frame is $\gamma \frac{\ell_0}{c}$. Therefore the distance from *S* to *Q* is $\gamma \ell_0$. If you read through that quickly, you're probably scratching your head. Go back over this argument carefully and be sure you understand it.

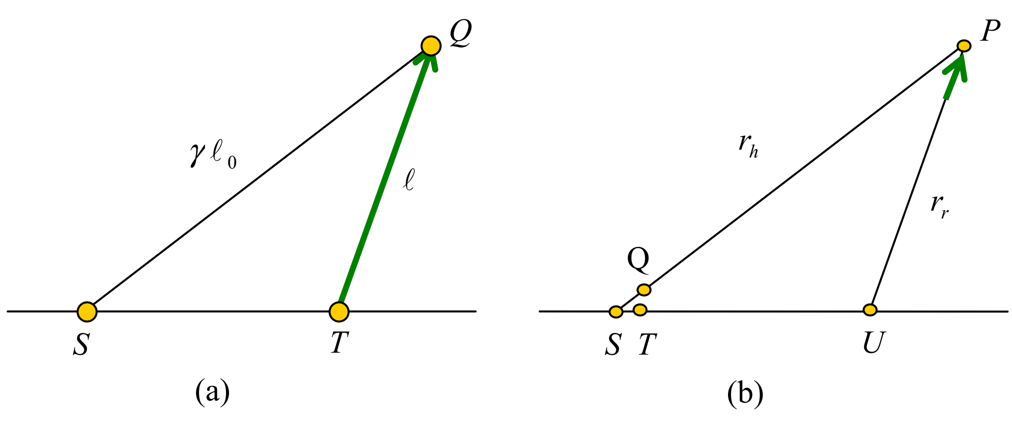

Figure 2.12. A thread emitted from source at *S* (a) just at the time the tail is emitted from the source and (b) at a later time when the thread reaches a point *P*. Note that the triangles are similar.

The length of the thread, though, is the distance from T to Q. To determine this length, we can compare the geometry of Fig. 2.12 (a) to that of the thread at a later time, Fig. 2.12 (b). We see that the thread has moved along the direction of the head line to a point P. We know that the ray from S to P is the head line and the ray U to P is the ray line. The triangles in figures are similar, so:

$$\frac{\ell}{\gamma \ell_0} = \frac{r_r}{r_h}$$

Since the thread is in the direction of the ray line, we can write this as a vector:

(2.3 Thread length, moving source) $$\vec{\ell} = \ell_0 \gamma \frac{\vec{r}_r}{r_h}$$

ℓ_0 is the length of threads emitted from a rest source in meters (m).

γ is defined by the equation $\gamma = 1/\sqrt{1-\beta_s^2}$, and in turn

β_s is the ratio of the velocity of the source to the speed of light, $\beta_s = v_s / c$.

$\vec{r}_r$ is the (vector) ray line in meters (m).

$\vec{r}_h$ is the (vector) head line in meters (m).

r_h is the magnitude of $\vec{r}_h$ in meters (m).

The formula for the density of the threads is a little more complicated to derive, so I'll just give it to you. If you'd like to see the details, a derivation is found in Appendix A. Some useful results proved there are:

(2.4)
$$\frac{r_r}{r_h} = \sqrt{1 - 2\beta\cos\theta + \beta^2}$$
$$\frac{r_0}{r_r} = \gamma\sqrt{1 - \beta^2 \sin^2 \psi}$$
$$\frac{r_h}{r_0} = \gamma(1 + \beta\cos\theta_0) = \frac{1}{\gamma(1 - \beta\cos\theta)}$$

The formula for the density of threads is:

(2.5 Thread density, moving source) $$\nu = \frac{N_0 q_s}{4\pi\, ec} \frac{r_h}{r_0^3} = \nu_0 \frac{r_h}{r_0}$$

- ν is the number density of threads in the vicinity of the field particle.
- N_0 is the total number of threads emitted per second by an electron at rest.
- q_s is the charge of the source particle in coulombs (*C*).
- r_h is the length of the head line in meters (*m*).
- r_0 is the distance from the source when the thread is emitted to the test charge in the rest frame of the source. Its units are meters (*m*).
- c is the speed of light, 2.998×10^8 *m/s*.
- ν_0 is the number density of threads in the vicinity of the test charge when the source is at rest.

As we learned in Chapter 1, the constants in these equations are related by the requirement that

$$N_e \ell_0 = c\,.$$

Combining Eqs. (2.2)-(2.5), we get:

$$\begin{aligned}\vec{F}_{thread} &= \frac{e}{\varepsilon_0} q_f \frac{\ell_0 \gamma \vec{r}_r}{r_h} \frac{N_0 q_s}{4\pi ec} \frac{r_h}{r_0^3} \\ &= \frac{q_s q_f}{4\pi\varepsilon_0} \frac{\gamma \vec{r}_r}{r_0^3}\end{aligned}$$

Finally, we use Eq. 2.4b to eliminate r_0 since we can't easily measure it.

The force on a stationary field particle from a moving source particle

(2.6 Thread force) $$\vec{F}_{thread} = \frac{q_s q_f}{4\pi\varepsilon_0} \frac{\vec{r}_r}{r_r^3 \gamma^2 (1-\beta_s^2 \sin^2\psi)^{3/2}}.$$

$\vec{F}$ is the force on a stationary field particle from a moving source particle in newtons (N).

ε_0 is 8.85×10^{-12} C^2/Nm^2.

q_s is the charge of the source particle in coulombs (C).

q_f is the charge of the field particle in coulombs (C).

γ is defined by the equation $\gamma = 1/\sqrt{1-\beta_s^2}$, and in turn

β_s is the ratio of the velocity of the source to the speed of light, $_s = v_s / c$.

$\vec{r}_r$ is the ray line in meters (m).

ψ is the angle the ray line makes with respect to the direction the source particle is moving.

Things to remember:
- By applying relativity to the motion of a source particle, we can calculate the length and the density of stubs produced by a moving source particle.
- You do **not** need to reproduce the arguments.
- Be able to use Eq. (2.6).

2.6 Stubs and the Stub Force

In this section, you will need to use cross products extensively. If you feel at all rusty, you might find it useful to read the section on cross products in the Vector Review before continuing.

Now we know how to find the force on a field particle from a moving source particle; however, the equation only works when the field particle is at rest. What we finally want to know is the force on the field particle when the field particle is moving as well as the source particle.

We do know, however, that we can always view the interaction from the rest frame of the field particle. If we do this, we can use the equations of the last section to find the force in this frame. Once we have found the force in the rest frame of the field particle, we can transform the force back into the original inertial frame where we began. What I will do is give you a geometrical construction based on the Thread Model that accomplishes this transformation is a rather simple way. A formal proof that this model is valid is given in Appendix C. This model is based on vectors called stubs that in many ways resemble threads.

Stubs

A stub is a short line segment that we attach to the head of every thread. If $\vec{\ell}$ is the thread, then the stub is

(2.7 Stub definition) $$\vec{s} = \hat{r}_h \times \vec{\ell}$$

$\hat{r}_h$ is the unit vector from the source (when the thread is emitted) to the head. That is, it is a unit vector in the direction of the head line. It is dimensionless.

$\vec{\ell}$ is the (vector) length of the thread. It has units of meters (m).

From this, we see that the length of stub is proportional to the length of the thread and to the sine of the angle between the head line and the thread. When the source is at rest, the head line and the thread are in the same direction (see Fig. 2.4), so there is no stub. The faster the source goes, the larger the angle between the head line and the thread, so the stub gets bigger. Also, at angles just in front and behind the particle, the angle between the head line and the thread is zero as can be seen in Fig. 2.1. Therefore, there are no stubs along the path of the particle.

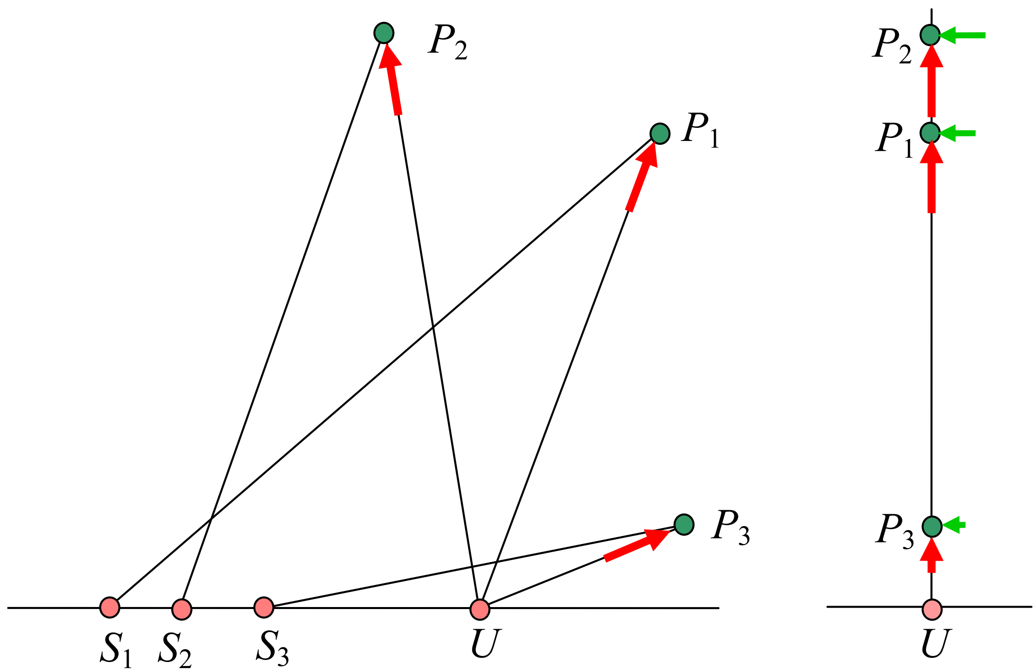

Figure 2.11. Threads and stubs of a positive charge. The threads (red) point away from the source. The stubs come out of the page. The figure on the right is a side view where the stubs (green) can be seen more clearly.

Figure 2.11 illustrates the geometry of threads and stubs in two dimensions. Threads emitted from a positive charge at three different times in the past arrive at points P_1, P_2, and P_3

at the same time that source has moved to point *U*. The threads point from *U* toward each field point. The stubs are given by the equation $\vec{s} = \vec{r}_h \times \vec{\ell}$. You may verify that the direction is out of the page in each case. When the thread and the head lines are nearly parallel, as in the case of P_3, the stubs are small. Figure 2.12 shows a similar diagram for a negatively charged source. Note that for positive charges, we draw both threads and stubs pointing toward the head, whereas for negative charges the threads and stubs both point away from the head.

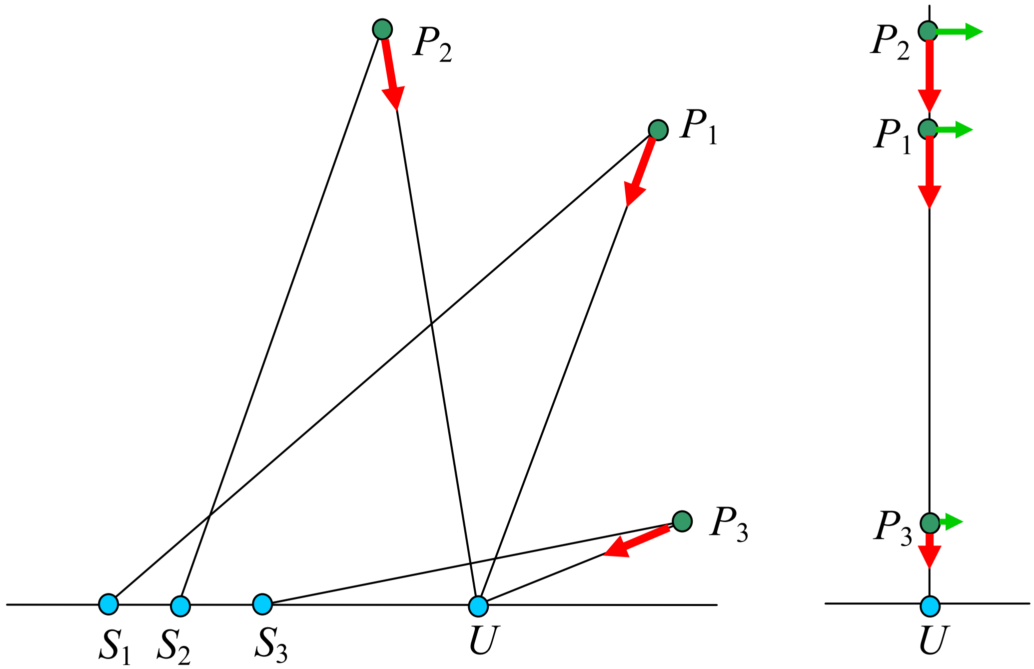

Figure 2.12. Threads and stubs of a negative charge. The threads (red) point toward the source. The stubs go into the page. The figure on the right is a side view where the stubs (green) can be seen more clearly.

Figure 2.13 shows a three-dimensional version of heads (blue cubes), head lines (yellow, very long lines), threads (red, longer lines on the heads), and stubs (green, shorter lines on the heads) for a positively-charged source moving at 70% of the speed of light. The source is the black diamond. Keep in mind that the threads and stubs for a positive charge are drawn so that they point toward the head. *It is very important to get a good feeling for the geometry of the head lines, threads, and stubs, so be sure to spend some time looking at the figures!*

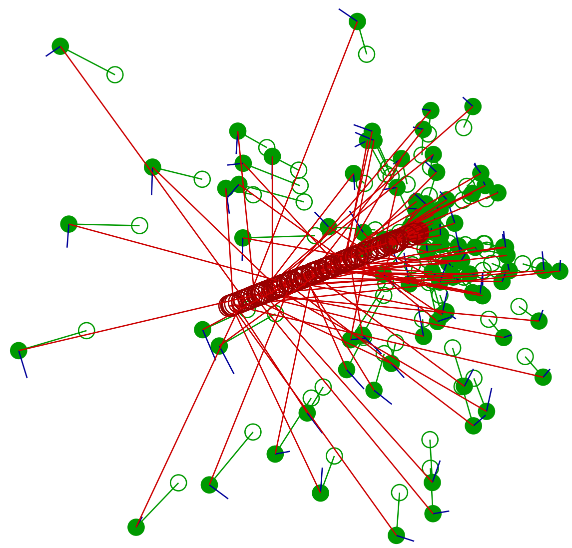

Figure 2.13. Threads and stubs of a moving positive charge with head lines.

I also have an animation of the threads and stubs of a source moving at 70% of the speed of light.

Uncompressed version:

http://www.physics.byu.edu/faculty/rees/220/AVIfiles/TS7.avi

At this point, you should have a fairly good idea of what stubs look like. However, you are still clueless as to what stubs do! The stubs give the relativistic corrections to the force for the motion of a field particle. The stub force is given by the relation:

Stub Force

(2.8 Stub force)
$$\vec{F}_{stub} = \frac{e}{\varepsilon_0} q_f v \left(\vec{\beta}_f \times \vec{s}\right).$$

where:

$\vec{s}$ is the stub (vector) in units of meters (m).

$\vec{\beta}_f$ is the (vector) velocity of the field particle divided by the speed of light. It is dimensionless.

the other variables are the same as in Eq. (2.2).

Note that the thread force and the stub force are very similar in form, except that the stub force is not proportional to the length of the stub, but to the cross product of the field particle velocity and the length of the stub. To visualize the stub force, we must consider this cross product. The force is small when the velocity of the field particle is small and it is small when the field particle is moving roughly parallel or antiparallel to the stub. Figures 2.14 and 2.15 are similar to Fig. 2.11 except that the velocity of the field particle $\vec{v}_f$ (pointing away from the head) is shown in black and the magnetic force is shown in cyan.

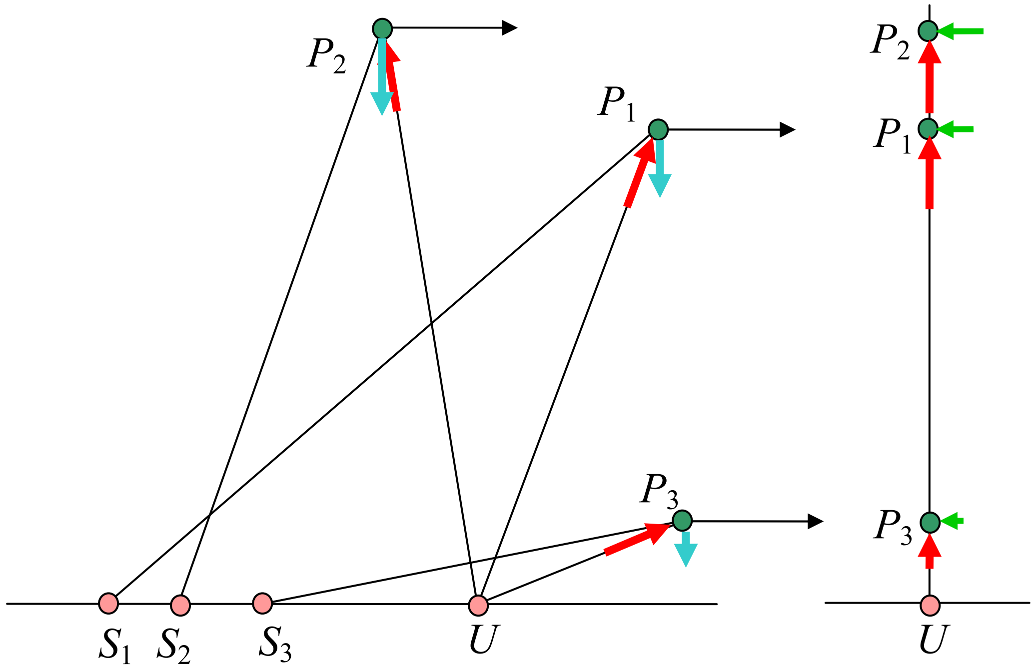

Figure 2.14. The magnetic force on moving field particles. The velocity of the field particles is indicated by the thin black arrows pointing left. The magnetic force is represented by the cyan arrows pointing downward.

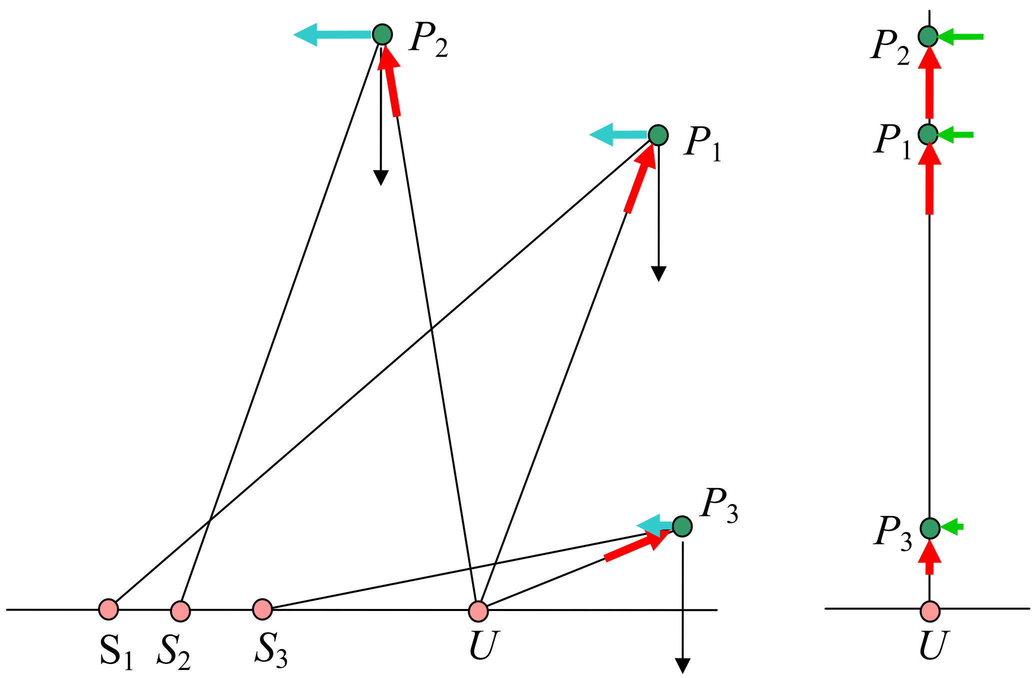

Figure 2.15. The same as Fig. 2.14 with the field particles moving downward.

Figure 2.16 shows a three dimensional view of the threads, stubs, and magnetic force. As in Fig. 2.14, the velocity of the field particle is shown as black arrow. Unlike the thread and stub of a positive source, the velocity arrow points away from the head.

Things to remember:
- Stubs are used to account for the motion of field particles.
- The definition of a stub is: $\vec{s} = \hat{r}_h \times \vec{\ell}$
- The stub causes a force that is proportional to the charge of the field particle, the thread density, and $\vec{\beta}_f \times \vec{s}$.

2.7 Electric and Magnetic Forces and Fields

The total force on the field particle is just the sum of the thread force and the stub force. Recapitulating:

(2.9 Thread force) $$\vec{F}_t = \frac{e}{\varepsilon_0} q_f \nu \vec{\ell}$$

(2.10 Stub force) $$\vec{F}_s = \frac{e}{\varepsilon_0} q_f \nu \left(\vec{\beta}_f \times \vec{s}\right)$$

The thread force is also known as the "electric force" and the stub force is known as the "magnetic force."

Without specifying the charge and the motion of the field particle we can define the thread and the stub, but we can't determine the forces on the field particle. We can think of threads and stubs as properties of the source. Force, however, depends on threads, stubs, and the charge and motion of the field particle. Traditionally physicists have separated the contributions of the source and field particles to define electric and magnetic fields. This is done as follows:

(2.11 Electric force) $$\vec{F}_t = \left(q_f\right)\left(\frac{e}{\varepsilon_0} \nu \vec{\ell}\right) = q_f \vec{E}_s$$

(2.12 Magnetic force) $$\vec{F}_s = \left(q_f \vec{\beta}_f\right) \times \left(\frac{e}{\varepsilon_0} \nu \vec{s}\right) = \left(q_f \vec{v}_f\right) \times \left(\frac{1}{c}\frac{e}{\varepsilon_0} \nu \vec{s}\right) = q_f \vec{v}_f \times \vec{B}_s$$

where

$\vec{E}_s$ is the electric field of the point charge in units of newtons/coulomb (*N/C*) or volts/meter (*V/m*).

$\vec{B}_s$ is the magnetic field of the point charge in units of tesla (*T*).

Often these forces are combined into a single force called the Lorentz force. The Lorentz force is named after Henrik A. Lorentz (1853 – 1928), a remarkable Dutch physicist who was an expert in electromagnetic theory. He derived the fundamental equations of special relativity a year before Einstein, but thought they could only apply to electromagnetic phenomena. *Be sure* *to memorize and understand the Lorentz force equation as it is one of the essential elements of this course.*

(2.13 The Lorentz force law) $$\vec{F} = q_f \vec{E}_s + q_f \vec{v}_f \times \vec{B}_s$$

There are a few things we should note about Eqs. (2.11)-(2.13).

- The electric and magnetic fields are defined in terms of forces.
- The electric and magnetic fields only depend on the source particles. The forces also depend on the charge and velocity of the field particles.
- There is no magnetic *field* when the source particle is at rest.
- The electric force on a positive field particle is in the same direction as the electric field. If the field particle is negative, the force is opposite the direction of the electric field.
- The magnetic force on a positive field particle is given by the right-hand rule for cross products. Note that it is always perpendicular to the plane formed by $\vec{v}_f$ and $\vec{B}$.
- There is no magnetic *force* when the field particle is at rest. That means that magnetic forces require motion of both source and field particles!
- The electric field of a point particle is related to the threads by the equation

(2.14) $$\vec{E} = \frac{e}{\varepsilon_0} \nu \vec{\ell}.$$

- The magnetic field of a point particle is related to the stubs by the equation

(2.15) $$\vec{B} = \frac{1}{c} \frac{e}{\varepsilon_0} \nu \vec{s}.$$

- Since there is one stub for every thread, the density ν is the same in both equations. Also note that the equation for the magnetic field has a factor of $1/c$ built into it, so magnetic fields are typically quite small in SI units.
- The units of electric field can be written in two equivalent ways: *N/C* or *V/m*.
- Since the lengths and densities of threads and stubs vary in time and vary from point to point in space, the electric and magnetic fields are functions of position and time.
- When there are many source particles, the total force on a field particle is the vector sum of all the forces from all of the sources. Equation (2.13) still holds, but the fields are the vector sums of the fields from each source particle.
- If we don't know detailed information about source particles, but know the forces on test particles, we can use Eq. (2.13) to determine the electric and magnetic fields.

Although Eqs. (2.11) and (2.12) are simple expressions for the electric and magnetic fields of moving point charges, it is more convenient to write the fields in terms of the quantities we can easily measure in the lab frame: $\vec{\beta}_s$, $\vec{\beta}_f$, $\vec{r}_r$ and ψ.

First, we recall that

$$\vec{s} = \hat{r}_h \times \vec{\ell}$$

so

$$\vec{\beta}_f \times \vec{s} = \vec{\beta}_f \times \left(\hat{r}_h \times \vec{\ell}\right).$$

To find $\hat{r}_h$, we turn to Fig. 2.1:

$$\vec{r}_h = r_h \vec{\beta}_s + \vec{r}_r$$

$$\hat{r}_h = \frac{\vec{r}_h}{r_h} = \vec{\beta}_s + \frac{\vec{r}_r}{r_h}$$

We note that the thread is in the ray direction so $\vec{\ell}$ is parallel (or antiparallel) to $\vec{r}_r$. Now, recalling that the cross product of two parallel vectors is zero, we can reduce the triple cross product above to:

$$\vec{\beta}_f \times \left(\hat{r}_h \times \vec{\ell}\right) = \vec{\beta}_f \times \left[\vec{\beta}_s \times \vec{\ell} + \frac{1}{r_h}\vec{r}_r \times \vec{\ell}\right] = \vec{\beta}_f \times \left(\vec{\beta}_s \times \vec{\ell}\right)$$

That means that the magnetic force is related to the electric force by the relation:

$$\vec{F}_B = \vec{\beta}_f \times \left(\vec{\beta}_s \times \vec{F}_E\right)$$

By applying the Lorenz force law, we can deduce one more relationship that is useful:

$$\vec{F}_B = \vec{\beta}_f \times \left(\vec{\beta}_s \times \vec{F}_E\right)$$

$$q_f \vec{v}_f \times \vec{B} = \vec{\beta}_f \times \left(\vec{\beta}_s \times q_f \vec{E}\right)$$

$$q_f \vec{\beta}_f c \times \vec{B} = \vec{\beta}_f \times \left(\vec{\beta}_s \times q_f \vec{E}\right)$$

$$\Rightarrow \vec{B} = \frac{1}{c}\vec{\beta}_s \times \vec{E}$$

In turn, we know the thread force from Eq. (2.6). Putting all of these together, we find that the fields of a non-accelerating point charge are:

Fields of Moving Point Charges

(2.16 Electric Field)
$$\vec{E} = \frac{q_s}{4\pi\varepsilon_0} \frac{\vec{r}_r}{r_r^3 \gamma_s^2 (1 - \beta_s^2 \sin^2 \psi)^{3/2}}$$

(2.17 Magnetic Field)
$$\vec{B} = \frac{1}{c} \vec{\beta}_s \times \vec{E}$$

$\vec{E}$ is the electric field of a source particle moving at constant velocity. It is in units of newtons/coulomb (*N/c*) or vols/meter (*V/m*).
$\vec{B}$ is the magnetic field of a moving source particle in units of tesla (*T*).
ε_0 is $8.85 \times 10^{-12}\ C^2/Nm^2$.
c is the speed of light, $2.998 \times 10^8\ m/s$.
q_s is the charge of the source particle in coulombs (*C*).
$\vec{\beta}_s$ is the ratio of the velocity of the source to the speed of light, $\vec{\beta}_s = \vec{v}_s / c$.
$\gamma_s = 1/\sqrt{1-\beta_s^2}$
$\vec{r}_r$ is the ray line in meters (*m*).
ψ is the angle the ray line makes with respect to the direction the source particle is moving.

These equations probably seem rather intimidating! But don't worry, they're not as hard as they seem. We'll see some examples of their use a little later in this chapter, but we won't need to use them extensively.

Things to remember:
• Electric and magnetic fields are understood in terms of forces. If a field particle is at rest, only electric forces act on the particles. When both the source particle and field particle are in motion, magnetic fields act on them.
• The Lorentz force equation is $\vec{F} = q_f \vec{E}_s + q_f \vec{v}_f \times \vec{B}_s$.
• The electric field depends on thread length and thread density. The magnetic field depends on stub length and thread (or stub) density.
• Know how to use Eqs. (2.16) and (2.17). You do not need to memorize these equations.

2.8 Visualizing Threads, Stubs, and Field Lines

By now you should be getting a feeling of the directions of threads and stubs. If we do a little sleight of hand, however, we can make the threads and stubs easier to visualize.

As far as forces are concerned, it doesn't really matter that the threads and stubs are themselves moving at the speed of light; the force is dependent only on the length of the thread, the density of the threads, and the velocities of the source and field particles. Furthermore, it doesn't really matter exactly where each thread is located as long as we don't change the density of threads within a region of interest. So if we move the threads around just a little and straighten them up in a more regular sort of pattern, it doesn't affect the results of the previous section at all. The sleight of hand we are going to use is this: we will move all the threads just a little so that the threads form into lines with their heads in rows. Figure 2.16 illustrates this process.

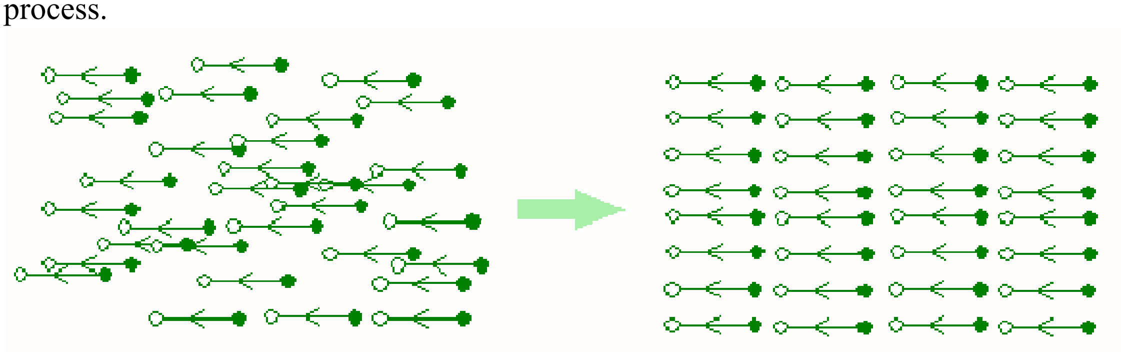

Figure 2.16. Aligning the threads.

We can also align the stubs in exactly the same way. Aligned threads are called "electric field lines." Aligned stubs are called "magnetic field lines." We will spend much of this course studying the characteristics of electric and magnetic field lines.

Now let's align the threads of a moving point charge. The threads and stubs of a point charge moving at 70% of the speed of light are depicted in Fig. 2.17. The image shows the threads and stubs at a given instant in time. At this time, the source particle is located at the vertex of the cones. We can summarize the systematics of the system in a few rules, some of which we already knew:

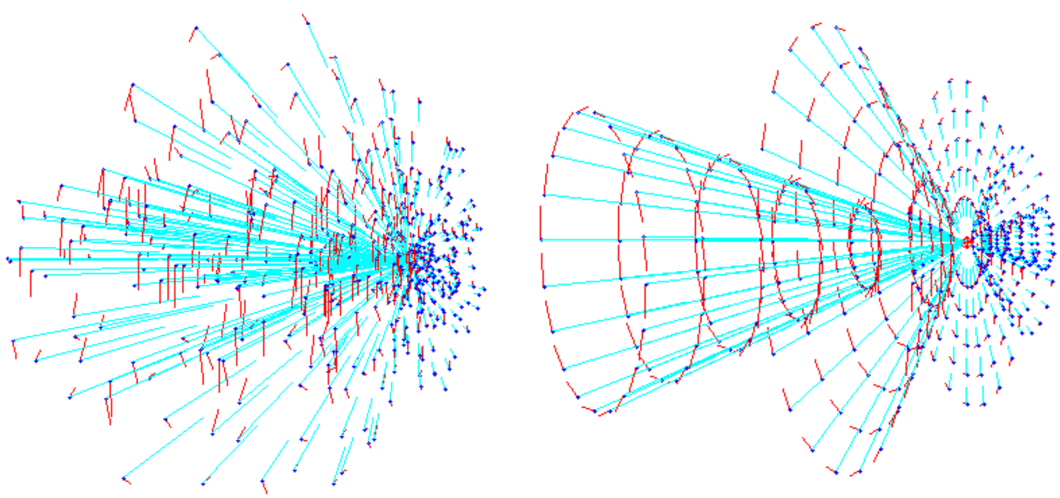

Figure 2.17. Aligning the threads of a moving charge.
The threads are cyan and the stubs are red.

Animations of threads and stubs being aligned can be found at:

http://www.physics.byu.edu/faculty/rees/220/AVIfiles/ThrAlign.avi
http://www.physics.byu.edu/faculty/rees/220/AVIfiles/StAlign.avi

Things to remember:

• The electric field lines point away from the source along the ray lines. If the source were negative, they would point back toward the source.
• There are more threads in the forward direction than in the backward direction.
• The threads are shorter in the forward direction than in the backward direction.
• The electric field lines are similar in the forward and backward directions, pointing out that the electric field is the same.
• The stubs are at right angles to the threads and form a circular magnetic field lines.
• The direction of the stub for a positive source particle is given by a right-hand rule. Point the thumb of your right hand in the direction of the velocity, and your fingers curl around in the direction of the stubs. If the source were negative, the stubs point in the opposite direction. This is also the direction of the magnetic field lines around the wire. (This is just a consequence of the definition $\vec{s} = \hat{r}_h \times \vec{\ell}$.)

2.9 Finding Forces of Moving Charges

With the help of Eqs. (2.16) and (2.17) and the Lorentz force law, we can, in principal, find the force on a field particle from any collection of source charges, as long as the source charges do not accelerate. In this section, we will begin by finding the force of a single moving source charge, and then see how integration can be used to find the force on a field particle from a proton beam and from long, straight current-carrying wire.

Example 2.1. The Force of a Single Moving Charge

Equations (2.16) and (2.17) are really quite general. Because of that, they're sometimes difficult to interpret. To actually use these equations, we must first set up a specific coordinate system. Since we have complete latitude in setting doing this, let's do it in a way that is as simple as possible. Consider the problem of a field particle located at a fixed point *P* and a source charge moving along a straight line. We wish to calculate the force on the field charge at a specific time. At this time, the source particle is at a position *U* along the line. Without any loss of generality, we can say the source particle moves in the +*x* direction. Then we can say that *P* lies on the *y* axis. This geometry is shown in Fig. 2.18.

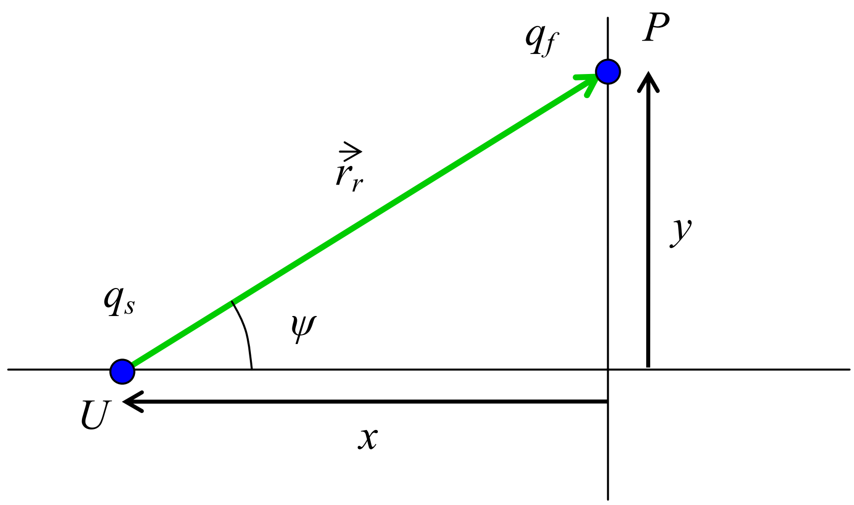

Figure 2.18. Setting up a coordinate axis for Eqs. (2.16) and (2.17).

To apply Eqs. (2.16) and (2.17), we must identify the three quantities $\vec{r}_r$, r_r, and $\sin\psi$ in the coordinate system we have chosen. By inspecting Fig. 2.18, we can conclude that

$$\vec{r}_r = -x\hat{x} + y\hat{y}$$
$$r_r = \sqrt{x^2 + y^2}$$
$$\sin\psi = \frac{y}{r_r}$$

In the first equation, there is a minus sign on the *x* because the *x* component of $\vec{r}_r$ is positive when *U* is to the left of the origin ($x<0$) and negative when *U* is to the right of the origin ($x>0$).

As long as we are given the charges of the two particles, the velocities of the two particles, and values for *x* and *y*, then we can find the force by simple substitution into this equation. We will also need the following relations in addition to Eqs. (2.16) and (2.17) to solve for the force:

$$\vec{\beta}_s = \frac{\vec{v}_s}{c}, \quad \vec{\beta}_f = \frac{\vec{v}_f}{c}$$
$$\gamma_s = \frac{1}{\sqrt{1-\beta_s^2}}$$
$$F = q_f\vec{E} + q_f\vec{v}_f \times \vec{B}$$

If we want to be more elegant, we can use algebra to simplify the denominator before we put in numbers. But this is just for elegance.

$$
\begin{aligned}
&r_r^3 \gamma_s^2 \left(1-\beta_s^2 \sin^2 \psi\right)^{3/2} = \\
&\left(r_r^2\right)^{3/2} \gamma_s^2 \left(1-\beta_s^2 \sin^2 \psi\right)^{3/2} = \\
&\gamma_s^2 \left(r_r^2 - \beta_s^2 r_r^2 \sin^2 \psi\right)^{3/2} = \\
&\gamma_s^2 \left(r_r^2 - \beta_s^2 y^2\right)^{3/2} = \\
&\gamma_s^2 \left(x^2 + y^2 - \beta_s^2 y^2\right)^{3/2} = \\
&\gamma_s^2 \left[x^2 + y^2 (1-\beta_s^2)\right]^{3/2} = \\
&\frac{1}{\gamma_s} \gamma_s^3 \left[x^2 + y^2 (1-\beta_s^2)\right]^{3/2} = \\
&\frac{1}{\gamma_s} \left[\gamma_s^2 x^2 + y^2\right]^{3/2} \qquad as\ \gamma_s^2 = \frac{1}{1-\beta_s^2}
\end{aligned}
$$

Now from Eqs. (2.16)–(2.17), we see that we have:

$$
\begin{aligned}
\vec{E} &= \frac{q_s}{4\pi\varepsilon_0} \frac{\gamma_s \left(-x\hat{x} + y\hat{y}\right)}{\left(\gamma_s^2 x^2 + y^2\right)^{3/2}} \\
\vec{B} &= \frac{1}{c} \vec{\beta}_s \times \vec{E} \\
F &= q_f \vec{E} + q_f \vec{v}_f \times \vec{B}
\end{aligned}
$$

The electric force lies along the ray line $\vec{r}_r$ and that the magnetic field is out of the page (in the $\vec{\beta}_s \times \vec{E}$ direction), as predicted by the "right-hand rule."

For the next example, we do something much harder: we find the force on a field particle from an infinite number of source particles. Of course when we talk about infinite sums, we know we're going to have to integrate. Notice that setting up the integral isn't really too hard, and thanks to computers, evaluating it isn't hard either. I will *not* expect you to evaluate integrals that arise in these problems, except in some cases where the integral is of a very simple function.

Example 2.2. The Force of a Proton Beam on a Field Particle

A certain particle accelerator creates a beam of protons that is used to probe the structure of matter. We wish to find the force on a field particle placed a distance *y* above the proton beam. We will assume that the field point *P* is in a region where the beam goes in a straight line a very long distance to both the right and to the left, as illustrated in Fig. 2.19. Furthermore, let us assume that the "linear charge density," the charge per unit length of the beam, is known.

(Note that it's impossible to define a total charge for an infinitely long beam!) We represent the linear charge density by λ which has units of coulombs per meter (C/m).

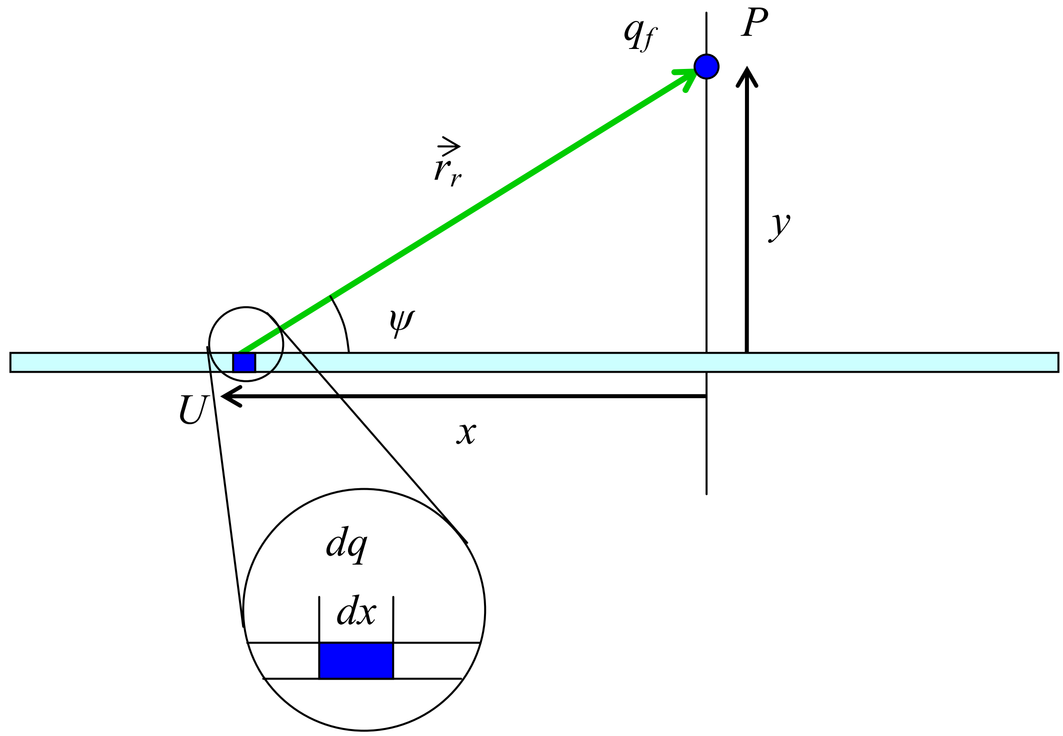

Figure 2.19. Calculating the force of a proton beam on a field particle.

One more approximation that we'll make is to treat the speed of the threads as effectively infinite. This allows us to ignore the fact that threads arriving at P at one time are emitted by source charges at many different times. The approximation is very good as long as the proton beam isn't going a significant fraction of the speed of light.

Eqs. (2.16) and (2.17) tell us the fields of point charges, but they can't be used directly when the source charge is an infinitely long line. We can, however, break down the proton beam into many little slices that are each essentially point charges. *This is an important trick that we use over and over in physics.* We can find the force from one of these little slices, and then add all these forces together to get the total force. The process may seem complicated, but this is precisely the sort of problem Newton and Leibniz invented calculus to solve.

First, we take one slice of the proton beam. The slice has length dx and is located at coordinate x along the x axis. The charge, dq, on this slice can be determined because we know the linear charge density:

$$\lambda = \frac{dq}{dx} \Rightarrow dq = \lambda\, dx\,.$$

The force on a field particle from this little slice is then just what we found in Example 2.1. We only have to change the notation a bit to remind us that the force is only the force from one little slice.

$$d\vec{E} = \frac{dq}{4\pi\varepsilon_0}\frac{\gamma_s(-x\hat{x}+y\hat{y})}{\left(\gamma_s^2 x^2+y^2\right)^{3/2}}$$

$$d\vec{B} = \frac{1}{c}\vec{\beta}_s \times d\vec{E}$$

$$dF = q_f d\vec{E} + q_f \vec{v}_f \times d\vec{B}$$

To find the total force, all we need to do is add up the contributions of all the slices that extend from $x=-\infty$ to $x=+\infty$.

$$\vec{F}_E = q_f\vec{E} = q_f\int d\vec{E}_t = q_f\int_{-\infty}^{+\infty}\frac{\lambda\, dx\,\gamma_s}{4\pi\varepsilon_0}\frac{(-x\hat{x}+y\hat{y})}{\left(\gamma_s^2x^2+y^2\right)^{3/2}}$$

$$\vec{F}_B = \int d\vec{F}_B = \int\vec{\beta}_f\times\left(\vec{\beta}_s\times d\vec{F}_E\right) = \vec{\beta}_f\times\left(\vec{\beta}_s\times\int d\vec{F}_E\right) = \vec{\beta}_f\times\left(\vec{\beta}_s\times\vec{F}_E\right)$$

$$\vec{F} = \vec{F}_E + \vec{F}_B$$

Let's first evaluate the electric force explicitly:

$$\vec{F}_E = -\hat{x}\frac{\lambda q_f\gamma_s}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{x}{\left(\gamma_s^2x^2+y^2\right)^{3/2}}dx + \hat{y}\frac{\lambda q_f\gamma_s y}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{1}{\left(\gamma_s^2x^2+y^2\right)^{3/2}}dx$$

The first integral must be zero because we have an odd integrand evaluated over a symmetric interval. If that rule isn't familiar to you, you need to learn it! The idea is simple. If we write the integrand as $I(x)$, then an odd integrand means that $I(-x)=-I(x)$. If the integrand is symmetric, it means the limits are from +*R* to –*R*. If this is true, the area underneath the curve along the negative *x* axis is opposite in sign to the area under the curve along the positive *x* axis. Hence, the integral of any odd integrand over symmetric limits is zero.

You probably did the second integral in your calculus class. It can be evaluated fairly easily by making the substitution $w = y\tan\gamma_s x$. However, I'm just going to tell you that the value of the integral is $2/y^2$. (In this class I won't worry about techniques for evaluating integrals, but I will expect you to be able to set up the integrals.)

(2.18) $$\vec{F}_E = \hat{y}\frac{\lambda q_f y}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{1}{\left(u^2+y^2\right)^{3/2}}du = \hat{y}\frac{\lambda q_f y}{4\pi\varepsilon_0}\frac{2}{y^2} = \frac{\lambda q_f}{2\pi\varepsilon_0 y}\hat{y}$$

We should note one important point about this result. The electric force has no dependence on the speed of the source. We did stipulate that the speed of the beam must be small compared to the speed of light; however, that just makes the calculation simpler without changing the result. A more general derivation is given in Appendix C. The conclusion is that the electric force on a charge near a proton beam is the same if the beam is nearly at rest or moving at close to the speed of light – as long as the number of protons per meter is the same in both cases.

Finally, we wish to calculate the magnetic force. This is:

$$\begin{aligned}\vec{F}_B &= \vec{\beta}_f \times \left(\vec{\beta}_s \times \vec{F}_E\right) \\ &= \vec{\beta}_f \times \left(\beta_s \hat{x} \times \frac{\lambda q_f}{2\pi\varepsilon_0 y}\hat{y}\right) \\ &= \vec{\beta}_f \times \left(\frac{\lambda q_f \beta_s}{2\pi\varepsilon_0 y}\hat{z}\right) \quad as \quad \hat{x}\times\hat{y} = \hat{z}\end{aligned} \tag{2.19}$$

Before we can finish the problem, we need to know the velocity of the field particle. We can make some important observations, though:

1) By the Lorentz force law, we know that the quantity in parentheses is *c* times the magnetic field and is proportional to the stub. Note that the direction of the field is consistent with the right hand rule or the cross product rule.
2) The maximum value of the magnetic force occurs when the angle between $\vec{\beta}_f$ and the stub is 90º.
3) The maximum magnitude of the magnetic force is $\beta_s\beta_f$ times the electric force. The magnetic force can never be larger than the electric force, and when particles are only going a small fraction of the speed of light, it is much smaller than the electric force.

Think About It

The result we have just obtained is only good for a field point on the +*y* axis. If the field particle were on the –*y* axis, what would the result be? How about the +*z* axis? the –*z* axis?

Example 2.3. The Force of a Current-Carrying Wire on a Field Particle

We can think of a wire as composed of four different components: the neutrons, the protons, the bound electrons, and the conduction electrons. The neutrons are electrically neutral and can be ignored. The protons and the bound electrons are fixed in space while the conduction electrons are moving along the wire at a given speed.

First, let us consider the electric forces. We found in the last problem that these forces are independent of the speed of a proton beam. In our wire, the thread forces are the same when the conduction electrons are moving and when the conduction electrons are at rest. If the conduction electrons are at rest, then the total number of positive charges in a given length of wire must equal the total number of negative charges in the same length of wire. Hence, the overall charge density in the wire is zero and the electric force is zero as well.

This means that the only force felt by a field particle is the magnetic force. Although the magnetic force is generally small compared to the electric force, in this case, the electric force cancels out. The magnetic force of a current-carrying wire is easily observed and can be put to many applications. In fact, you have probably made electromagnets yourself.

To quantify these results, we can borrow the conclusion of the last example:

$$\vec{F}_B = \vec{\beta}_f \times \left(\frac{\lambda q_f \beta_s}{2\pi\varepsilon_0 y} \hat{z} \right)$$

where λ is the charge density of *conduction* electrons in the wire. Note that since λ is negative, the stubs at point *P* are in the $-z$ direction.

Now, let's make the example concrete: A copper wire has a diameter of $d = 1.00$ mm. A current of I = 2.50 amperes (1 ampere = 1 coulomb/second) passes through the wire. The electrons in the wire are moving with an average speed of $v = 2.35\times10^{-4}$ m/s. An electron outside the wire moves parallel to the conduction electrons at a distance of 2.00 mm from the center of the wire at 30% of the speed of light. Find the force on the electron.

Some useful information:
mass of an electron: $m_e = 9.11\times10^{-31}$ kg
density of copper: $\rho = 8940$ kg/m^3
atomic weight of copper: $W = 0.063546$ kg/mole
Avagadro's number: $N_A = 6.022\times10^{23}$ atoms/mole
number of conduction electrons per atom of copper: $N_c = 1$
$\varepsilon_0 = 8.854\times10^{-12}\ C^2/Nm^2$

$$\beta_s = \frac{v}{c} = \frac{2.35\times10^{-4}}{2.998\times10^{8}} = 7.82\times10^{-13}$$

$$\#\text{ of conduction e}^- \text{ per unit volume} = \frac{N_c \times N_A \times \rho}{W} = \frac{1\times6.022\times10^{23}\times8920}{0.063546} = 8.473\times10^{28}\ e^-/m^3$$

$$\text{charge of conduction e}^- \text{ per unit length} = \lambda = 8.473\times10^{28}\ e^-/m^3 \times e \times \pi r^2$$

$$\lambda = 8.473\times10^{28} \times 1.602\times10^{-19} \times \pi \times .0005^2 = 10660\ C/m$$

That's a lot of charge!

$$F_B = \frac{\lambda q_f \beta_f \beta_s}{2\pi\varepsilon_0 y} = \frac{1.0660\times1.602\times10^{-19}\times0.3\times7.82\times10^{-13}}{2\pi\times8.854\times10^{-12}\times0.002} N = 3.60\times10^{-15} N$$

Things to remember:
• Be able to reproduce the examples in this section. This may be the hardest thing you have to do for the entire course! I do **not** expect you to do any other similar problems at this point – these examples are hard enough. But try to learn the examples with understanding, don't just memorize the steps.

2.10 The Magnetic Force and Relativity: An Intuitive Example

To understand how it is that relativity gives rise to the magnetic force, it is useful to look at one specific example. An infinitely long positively-charged rod is located along the *y* axis of a coordinate system, as shown in Fig. 2.20. Initially, at time t_0, a positive field particle is traveling in the *y* direction. The field particle feels a force in the *x* direction, and hence gains an *x* component to its velocity. The trajectory of the field particle as well as its location and velocity at a later time, t_1, are also indicated in the figure. In Newtonian mechanics we predict that the *y* component of the velocity would remain constant because there is no component of the force in that direction. Now, here's where relativity comes in. We know that the mass of an object increases as its velocity increases. We can not use $F=ma$ for a system where mass changes, but rather, we must use the more general relationship $F=dp/dt$. Since there is no component of the force in the *y* direction, the *y* component of momentum is constant. But if the mass is increasing, the *y* component of the velocity must be decreasing along the trajectory of the field particle. That is, $v_y < v_0$.

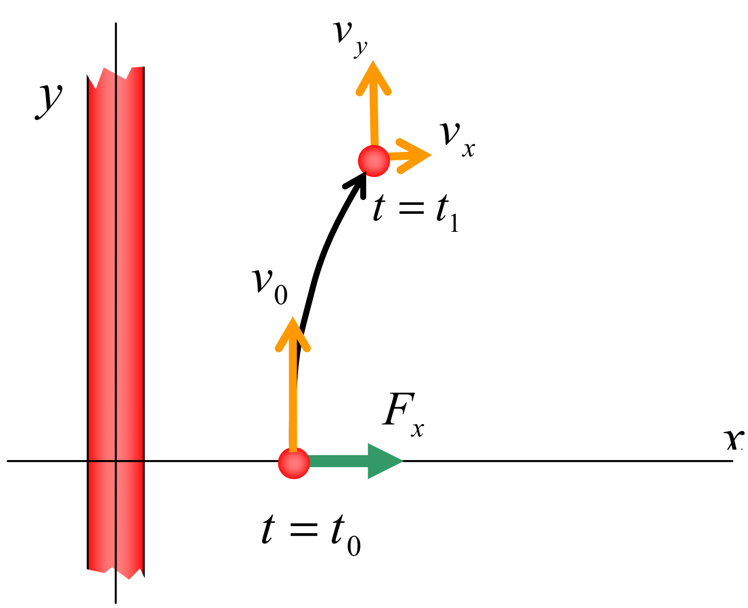

Fig. 2.20. The trajectory of a moving field particle being repelled by a long charged rod.

Let us now look at the same process, but this time from the viewpoint of an observer moving in the *y* direction with a velocity v_0. To this observer, the test charge appears initially to be at rest. In the course of time, the test charge must follow a trajectory like that shown in Fig. 2.21. The key idea here is that since the y component of the velocity is decreasing in Fig. 2.20, the field particle must move in the $-y$ direction in Fig. 2.21. Because of that, the observer concludes that a force in the $-y$ direction must exist for the field particle to accelerate downward. The observer in Fig. 2.21 attributes this part of the force to a magnetic field produced by the moving rod.

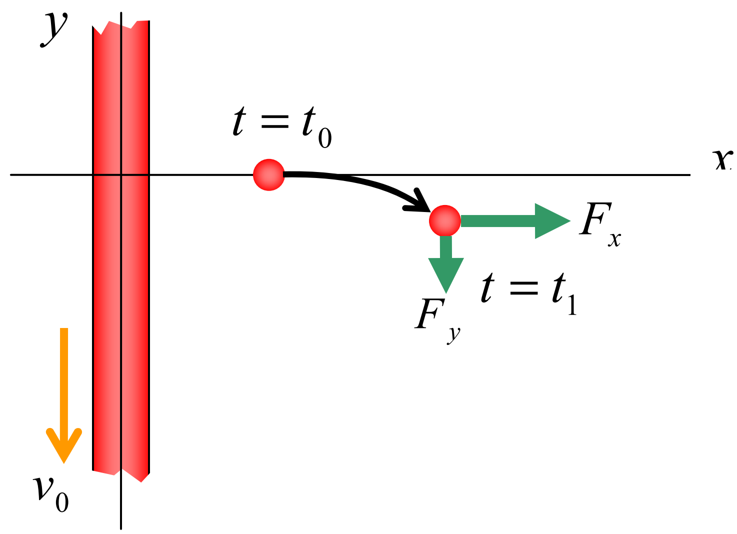

Fig.2.21. The trajectory of the field particle as viewed by an observer moving in the $+y$ direction.

If the rod is positively charged and moving downward, we can think of the rod as current going downward. We know that such a current produces stubs that circulate around the rod in the direction given by the right-hand rule. In this case, the stubs come out of the page in the region where the field particle is located. As the field particle accelerates to the right, the velocity gains a component in the $+x$ direction, which in turn produces a force is in the $-y$ direction. Thus we see that the magnetic force allows us to properly account for relativistic motion corrections when both the source charges and the field particle are in motion.

Things to remember:

- If an object moving in the y direction experiences a force in the x direction, the y component of its momentum remains unchanged. Since its mass increases, the y component of its velocity decreases. This is called "mass braking."
- If the motion of this same particle is viewed from the frame in which the particle is initially at rest, the particle accelerates in the $-y$ direction. In this frame, we have to attribute the motion to a velocity-dependent force. This force is the "stub" or "magnetic" force.