Complex Numbers Summary, by Dr Colton
Physics 471 – Optics

We will be using complex numbers as a tool for describing electromagnetic waves. P&W has a short section in Chapter 0 on the fundamentals of complex numbers, section 0.2, but here is my own summary.

Colton’s short complex number summary:

- A complex number $x + iy$ can be written in rectangular or polar form, just like coordinates in the $x$-$y$ plane.
  - The rectangular form is most useful for adding/subtracting complex numbers.
  - The polar form is most useful for multiplying/dividing complex numbers.
- The polar form $(A, \theta)$ can be expressed as a complex exponential $Ae^{i\theta}$.
- For example, consider the complex number $3 + 4i$:
  - $= (3, 4)$ in rectangular form,
  - $= (5, 53.13^\circ)$ in polar form, and
  - $= 5e^{i53.13^\circ}$ or $5e^{0.9273i}$ in complex exponential form, since $53.13^\circ = 0.9273$ rad.
- The complex exponential form follows directly from Euler’s equation: $e^{\theta} = \cos \theta + i \sin \theta$, and by looking at the $x$- and $y$-components of the polar coordinates.
- By the rules of exponents, when you multiply/divide two complex numbers in polar form, $(A_1, \theta_1)$ and $(A_2, \theta_2)$, you get:
  - multiply: $A_1e^{i\theta_1} \times A_2e^{i\theta_2} = A_1A_2e^{i(\theta_1+\theta_2)} = (A_1A_2, \theta_1+\theta_2)$
  - divide: $A_1e^{i\theta_1} \div A_2e^{i\theta_2} = (A_1/A_2)e^{i(\theta_1-\theta_2)} = (A_1/A_2, \theta_1-\theta_2)$
- I like to write the polar form using this notation: $A \angle \theta$. The “$\angle$” symbol is read as, “at an angle of”. Thus you can write:
  - $(3 + 4i) \times (5 + 12i)$
  - $= 5 \angle 53.13^\circ \times 13 \angle 67.38^\circ$
  - $= 65 \angle 120.51^\circ$ (since $65 = 5 \times 13$ and $120.51^\circ = 53.13^\circ + 67.38^\circ$)

Representing waves as complex numbers:

Suppose you have an electromagnetic wave traveling in the $z$-direction and oscillating in the $y$-direction. The equation for the wave would be this:

$$E = E_0 \hat{y} \cos(kz - \omega t + \phi)$$

It’s often helpful to represent that type of function with complex numbers, like this:

$$E = E_0 \hat{y} \cos(kz - \omega t + \phi) \rightarrow E = E_0 \hat{y}e^{i(kz-\omega t+\phi)}$$

It’s understood that this is just a temporary mathematical substitution. If you want to know the real oscillation, you take the real part of the complex exponential, i.e. turn it back into a cosine.

$$\rightarrow E = E_0 e^{i\phi} \hat{y}e^{i(kz-\omega t)}$$
$$\rightarrow E = E_0 \hat{y}e^{i(kz-\omega t)}$$

Now $E_0$ is actually a complex number whose magnitude is $E_0$, the wave’s amplitude, and whose phase is $\phi$, the phase of the oscillating cosine wave. This type of trick will make the math much easier for some calculations we need to do.