Cylindrical Waveguides
by Dr. Colton, Physics 442 (last updated: Winter 2020)

Background

To consider the case of cylindrical waveguides, i.e. formed by a hollow cylinder of radius \( R \), we again assume that the \( z- \) and \( t- \)dependence will be given by \( e^{i(kz-\omega t)} \). This leads to the same result from the wave equation as with a rectangular waveguide, only expressed in cylindrical coordinates. The equations for \( E_z \) and \( B_z \) are therefore as follows:

\[
\begin{align*}
\frac{\partial^2 E_z}{\partial s^2} + \frac{1}{s} \frac{\partial E_z}{\partial s} + \frac{1}{s^2} \frac{\partial^2 E_z}{\partial \theta^2} + \left( \frac{\omega^2}{c^2} - k^2 \right) E_z &= 0 \\
\frac{\partial^2 B_z}{\partial s^2} + \frac{1}{s} \frac{\partial B_z}{\partial s} + \frac{1}{s^2} \frac{\partial^2 B_z}{\partial \theta^2} + \left( \frac{\omega^2}{c^2} - k^2 \right) B_z &= 0
\end{align*}
\]

TM modes, Separation of Variables

For the TM modes, we have \( B_z = 0 \) and \( E_z \neq 0 \). We therefore focus on \( E_z \). Using the separation of variables technique, we assume that the solution has the form \( E_z = S(s)\Phi(\theta) \). This turns the equation for \( E_z \) into:

\[
S''\Phi + \frac{1}{s} S'\Phi + \frac{1}{s^2} S\Phi'' + \left( \frac{\omega^2}{c^2} - k^2 \right) S\Phi = 0
\]

Dividing both sides by \( S\Phi \) and multiplying by \( s^2 \), we have:

\[
s^2 \frac{S''}{S} + \frac{S'}{S} + \frac{\Phi''}{\Phi} + s^2 \left( \frac{\omega^2}{c^2} - k^2 \right) = 0
\]

Now bring the \( \theta \) term over to the right hand side, and we have successfully separated the variables.

\[
s^2 \frac{S''}{S} + \frac{\Phi''}{\Phi} = - \frac{\Phi''}{\Phi}
\]

The left hand side is just a function of \( s \), the right hand side is just a function of \( \theta \), so they can only be equal if they both equal a constant. It could be a positive or a negative constant, but because I know the answer I will guess correctly and make it a positive constant. To enforce that, we set it equal to \( \alpha^2 \).

\[
s^2 \frac{S''}{S} + \frac{\Phi''}{\Phi} = - \frac{\Phi''}{\Phi} = \alpha^2
\]

This is actually two equations, one for \( s \) and one for \( \theta \).

\[
s^2 \frac{S''}{S} + \frac{\Phi''}{\Phi} = \alpha^2
\]
$$\frac{\Phi''}{\Phi} = -\alpha^2$$

**Solving the Φ equation**

Let’s solve the φ equation first. It’s easy! \( \Phi'' = -\alpha^2 \Phi \) means that

$$\Phi = \begin{cases} \sin \alpha \phi \\ \cos \alpha \phi \end{cases}$$

or linear combinations.

Because \( \Phi(\phi) \) and \( \Phi(\phi + 2\pi) \) need to give the same value, this gives an added constraint that:

$$\alpha = \text{integer}$$

If that’s not obvious to you, try for example setting \( \phi = 30^\circ \) and comparing \( \sin(\alpha(30^\circ)) \) to \( \sin(\alpha(30^\circ + 360^\circ)) \) when \( \alpha \) is not an integer.

We can rotate the x- and y-axes such that we only get the cosine function. End result for \( \phi \), not including an arbitrary amplitude:

$$\Phi = \cos \alpha \phi$$

**Solving the S equation**

Now back to the S equation…

$$s^2 \frac{S''}{S} + s \frac{S'}{S} + s^2 \left( \frac{\omega^2}{c^2} - k^2 \right) = \alpha^2$$

$$s^2 S'' + s S' + s \left( s^2 \left( \frac{\omega^2}{c^2} - k^2 \right) - \alpha^2 \right)$$

Consider the term \( \frac{\omega^2}{c^2} - k^2 \). It has units of \((1/\text{length})^2\). By multiplying it by \( R^2 \), we can turn it into a dimensionless number. For reasons that will soon become clear, I’ll call that number \( u_{am}^2 \), so

$$u_{am} = R \sqrt{\frac{\omega^2}{c^2} - k^2}$$

That also means that:
Plugging that substitution for \( u_{am} \) back into the \( S \) equation, it turns the equation into this:

\[
s^2 S'' + s S' + S \left( \left( \frac{u_{am} S}{R} \right)^2 - \alpha^2 \right) = 0
\]

This is Bessel’s equation! Written for \( x = \frac{u_{am} S}{R} \) (Here \( x \) is a dimensionless variable, not the \( x \)-coordinate.)

Its solutions are the Bessel functions:

\[
S = \left\{ J_\alpha(x), Y_\alpha(x) \right\}
\]

or linear combinations.

The \( J_\alpha(x) \) functions are the regular Bessel functions. The \( Y_\alpha(x) \) functions are the “Bessel functions of the second kind”, which go infinite at the origin (\( s = 0 \)). Since we don’t want solutions which are infinite at the origin, we throw them out, leaving us the end result for \( s \), not including an arbitrary amplitude:

\[
S = J_\alpha \left( \frac{u_{am} S}{R} \right)
\]

**Putting together \( \phi \) and \( s \) solutions**

Putting the solutions together, \( E_z = S(s) \Phi(\phi) \), and still not worrying about an arbitrary amplitude, the answer is therefore:

\[
E_z = J_\alpha \left( \frac{u_{am} S}{R} \right) \cos \alpha \phi
\]

There could also be a summation over \( \alpha \); however usually people just consider each \( \alpha \) separately, as done below.

**Boundary conditions**

The governing boundary conditions are these two, evaluated at \( s = R \).

\[
E_{//} = 0
\]

\[
B_\perp = 0
\]

For the TM modes, we must focus on the \( E_{//} \) boundary condition. \( E_z \) is actually the parallel component, so it means \( E_z = 0 \).
We see now that the \( u_{\alpha m} \) values defined earlier must be the zeros of the Bessel functions.

**Some sample modes**

\( \alpha = 0 \)

\[ E_z = J_0 \left( \frac{u_{0m} S}{R} \right) \]

\( E_z \) has no \( \phi \) dependence. \( m \) can be any integer, and \( u_{0m} \) is the \( m \text{th} \) zero of the \( J_0 \) Bessel function. The possible modes are TM\(_{01} \), TM\(_{02} \), TM\(_{03} \), etc.

\( \alpha = 1 \)

\[ E_z = J_1 \left( \frac{u_{1m} S}{R} \right) \cos \phi \]

\( E_z \) does have \( \phi \) dependence now. \( m \) can be any integer, and \( u_{1m} \) is the \( m \text{th} \) zero of the \( J_1 \) Bessel function. The possible modes are TM\(_{11} \), TM\(_{12} \), TM\(_{13} \), etc.

Hopefully extrapolations to higher \( \alpha \) values are clear.

**Conclusion**

The allowed TM modes (\( B_z = 0 \)) are characterized by integer values for \( \alpha \) and \( m \). For a given mode, \( E_z = J_\alpha \left( \frac{u_{\alpha m} S}{R} \right) \cos \alpha \phi \). From \( E_z \), one can deduce all of the other components of the electric and magnetic fields using the “longitudinal to transverse” equations, if one desires. And the dispersion equation of a given mode is given by:

\[ k = \sqrt{\frac{\omega^2}{c^2} - \frac{u_{\alpha m}^2}{R^2}} \]

where (for the TM modes), \( u_{\alpha m} \) is the \( m \text{th} \) zero of the \( J_\alpha \) Bessel function.