Calculating TM Modes

Using Mathematica, we can calculate the first 16 TM modes for a rectangular waveguide. I’m using a dimensions of \( R = 10 \) cm which was chosen arbitrarily.

Here are the cutoff frequencies of the first 16 modes; they are shown first in table form and then in list form in ascending order.

\[
\begin{align*}
\text{In}[44] &= \ \text{umn}[\alpha, \ n_] = \text{BesselJZero}[\alpha, \ n]; \\
\text{cutoff}[\alpha, \ n_] &= \text{umn}[\alpha, \ n] \frac{c}{R} \\
\text{cutoffTable} &= \text{Table}[\text{cutoff}[\alpha, \ n], \{\alpha, 0, 3\}, \{n, 1, 4\}]; \\
\text{cutoffTable} &= \text{Flatten} // \text{Sort}
\end{align*}
\]

\[
\text{Out}[49]\text{MatrixForm} =
\begin{bmatrix}
7.21448 \times 10^9 & 1.65602 \times 10^{10} & 2.59612 \times 10^{10} & 3.53746 \times 10^{10} \\
1.14951 \times 10^{10} & 2.10468 \times 10^{10} & 3.05204 \times 10^{10} & 3.99711 \times 10^{10} \\
1.54069 \times 10^{10} & 2.52517 \times 10^{10} & 3.48595 \times 10^{10} & 4.43879 \times 10^{10} \\
1.91405 \times 10^{10} & 2.92831 \times 10^{10} & 3.90456 \times 10^{10} & 4.86704 \times 10^{10}
\end{bmatrix}
\]

\[
\text{Out}[50] = \{7.21448 \times 10^9, 1.14951 \times 10^{10}, 1.54069 \times 10^{10}, 1.65602 \times 10^{10}, 1.91405 \times 10^{10},
2.10468 \times 10^{10}, 2.52517 \times 10^{10}, 2.59612 \times 10^{10}, 2.92831 \times 10^{10}, 3.05204 \times 10^{10},
3.48595 \times 10^{10}, 3.53746 \times 10^{10}, 3.90456 \times 10^{10}, 3.99711 \times 10^{10}, 4.43879 \times 10^{10},
4.86704 \times 10^{10}\}
\]

The \( k(\omega) \) dispersion relations for the first 16 modes are as follows:

\[
\begin{align*}
\text{In}[15] &= k[w, \alpha, \ n_] := \sqrt{w^2/c^2 - \text{cutoff}[\alpha, \ n]^2/c^2} \\
\text{Table}[\text{K}[w, \alpha, \ n], \{\alpha, 0, 3\}, \{n, 1, 4\}] &= \text{Flatten} // \text{Sort} // \text{Reverse}
\end{align*}
\]

\[
\text{Out}[15] = \\
\begin{bmatrix}
\sqrt{\frac{-578.319}{900000000000000000000}}, \sqrt{\frac{-1468.2}{900000000000000000000}}, \\
\sqrt{\frac{-2637.46}{900000000000000000000}}, \sqrt{\frac{-3047.13}{900000000000000000000}}, \\
\sqrt{\frac{-4078.65}{900000000000000000000}}, \sqrt{\frac{-4921.85}{900000000000000000000}}, \\
\sqrt{\frac{-7488.7}{900000000000000000000}}, \sqrt{\frac{-9527.76}{900000000000000000000}}, \\
\sqrt{\frac{-13582.1}{900000000000000000000}}, \sqrt{\frac{-13984}{900000000000000000000}}, \\
\sqrt{\frac{-17752.1}{900000000000000000000}}, \sqrt{\frac{-21892}{900000000000000000000}}, \\
\sqrt{\frac{-26328.1}{900000000000000000000}}, \sqrt{\frac{-26389.9}{900000000000000000000}}, \\
\end{bmatrix}
\]
**$k(\omega)$ dispersion relation plots**

For a given mode its dispersion relation is set by one of the following curves.

For a given mode its dispersion relation is set by these curves. Note that these are the FIRST 16 modes, in the sense that $\alpha$ goes from 0 to 3 and $n$ goes from 1 to 4, but they are not necessarily the LOWEST 16 modes. For example, the $(\alpha = 4, n = 1)$ mode is lower than many of these that are shown (with its $\omega_{cutoff} = 2.28 \times 10^{10}$ rad/s).
Recall that the governing field for the TM modes is the $z$ component of the electric field (because the magnetic field has no $z$-component). Here are plots of $E_z$ for the first 16 modes. Aside from the upper left one, which has a node at the boundary and a single antinode in the middle, tannish white is the positive antinode and blue is the negative antinode.

$B_z = 0$ by definition, and all of the other nonzero components of the fields, namely $E_x, E_y, B_x,$ and $B_y,$ can be calculated from $E_z.$