theory of quantum electrodynamics, without making use of the methods of field quantization. Central to this presentation is the intuitive physical picture of particle and antiparticle propagation developed by Stückelberg and Feynman: particles propagate forward in time; antiparticles are viewed as particles with negative energy that move backward in time.

4.6 Supplement: The Δ Functions

In the last sections, several important functions have been constructed, starting from a scalar field. Now we want to present a complete collection of these functions and to derived their mutual relationships. The whole family of functions will be denoted by the letter Δ, each member being distinguished by an additional subscript. Table 4.1 contains a listing of the Δ functions commonly used.14

Table 4.1. A collection of the invariant commutation and propagation functions of a scalar field

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Contour</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δ(x)</td>
<td>Pauli-Jordan function</td>
<td>C</td>
<td>−</td>
</tr>
<tr>
<td>Δ(+)(x)</td>
<td>positive frequency</td>
<td>C(+)(x)</td>
<td>½(Δ + Δ1)</td>
</tr>
<tr>
<td>Δ(−)(x)</td>
<td>negative frequency</td>
<td>C(−)(x)</td>
<td>½(Δ − Δ1)</td>
</tr>
<tr>
<td>Δ1(x)</td>
<td>anticommutator</td>
<td>C1</td>
<td>−</td>
</tr>
<tr>
<td>ΔF(x)</td>
<td>Feynman propagator</td>
<td>C_F</td>
<td>½(ε(x_0)Δ + Δ1)</td>
</tr>
<tr>
<td>ΔR(x)</td>
<td>retarded propagator</td>
<td>C_R</td>
<td>Θ(x_0)Δ</td>
</tr>
<tr>
<td>ΔA(x)</td>
<td>advanced propagator</td>
<td>C_A</td>
<td>−Θ(−x_0)Δ</td>
</tr>
<tr>
<td>ΔD(x)</td>
<td>Dyson propagator</td>
<td>C_D</td>
<td>½(ε(x_0)Δ − Δ1)</td>
</tr>
<tr>
<td>Δ(x)</td>
<td>principal-part propagator</td>
<td>C</td>
<td>½ε(x_0)Δ</td>
</tr>
</tbody>
</table>

---

14 The literature contains a multitude of different conventions that differ in the names and by multiplicative factors.
These functions can be classified according to the result obtained when applying the Klein–Gordon operator. The functions \( \Delta_\Omega, \Delta^{(\tau)}, \Delta^{(-)}, \Delta_1 \) satisfy the *homogeneous Klein–Gordon equation*

\[
(\Box + m^2) \Delta(x) = 0 \quad \text{etc.} .
\]

(4.140)

In contrast, the functions \( \Delta_F, \Delta_R, \Delta_A, \Delta_D, \Delta \) are Green’s functions, i.e., they solve the *inhomogeneous Klein–Gordon equation* with a delta function as the source term:

\[
(\Box + m^2) \Delta_F(x) = -\delta^4(x) \quad \text{etc.} .
\]

(4.141)

In the following, we will call these two classes the *commutation functions* and the *propagation functions*.

**The Commutation Functions**

As the most important representative of this type, in Sect. 4.4 we have become acquainted with the *Pauli–Jordan function*, which was defined as the commutator of two scalar field operators:

\[
i \Delta(x) = [\hat{\phi}(x), \hat{\phi}(y)].
\]

(4.142)

The Pauli–Jordan function has the three-dimensional Fourier representation (using \( p_0 = \omega_p \equiv +\sqrt{p^2 + m^2} \))

\[
\Delta(x) = -i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left( e^{-ip\cdot x} - e^{+ip\cdot x} \right)
\]

(4.143)

or

\[
\Delta(x) = -\int \frac{d^3p}{(2\pi)^3} \epsilon^{ip\cdot x} \frac{\sin \omega_p x_0}{\omega_p} .
\]

(4.144)

Equivalently we have deduced the four-dimensional Fourier representation

\[
\Delta(x) = -i \int \frac{d^4p}{(2\pi)^4} \epsilon^{ip\cdot x} \epsilon(p_0) \delta(p^2 - m^2) ,
\]

(4.145)

where \( \epsilon(p_0) = \text{sgn}(p_0) \) denotes the sign function.

There is another way to represent \( \Delta(x) \) in terms of a four-dimensional integral in momentum space. Both terms in (4.143) can be interpreted as the residues of an integral over \( dp_0 \) having poles at \( p_0 = \pm \omega_p \). To generate these residues one has to integrate over a closed contour \( C \) that encompasses both poles (cf. Fig. 4.4a).

This is confirmed using the theorem of residues for the \( p_0 \) integration according to

\[
\int_C \frac{dp_0}{2\pi} \frac{e^{-ip_0x_0}}{p^2 - m^2} = \int_C \frac{dp_0}{2\pi} \frac{e^{-ip_0x_0}}{p_0^2 - \omega_p^2} \\
= -2\pi i \left( \text{Res}_{p_0 = +\omega_p} + \text{Res}_{p_0 = -\omega_p} \right) \\
= -i \frac{1}{2\omega_p} \left( e^{-i\omega_p x_0} - e^{+i\omega_p x_0} \right) .
\]

(4.146)

A comparison with (4.143) reveals that
\[ \Delta(x) = \int_{C} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2} , \quad (4.147) \]

where in the second term of (4.143) \( p \) was replaced by \(-p\).

The Pauli–Jordan function contains two parts, one with positive and one with negative frequency:

\[ \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) . \quad (4.148) \]

Obviously each of these parts is generated from one of the two residues in the integral (4.146). One can obtain these parts by using the contours \( C^{(+)} \) or \( C^{(-)} \), which circle around one of the poles as defined in Fig. 4.4b. The functions \( \Delta^{(\pm)}(x) \) can be written in the invariant form (4.147) by using the identities (see (4.101))

\[ \frac{1}{2\omega_p} \delta(p_0 \mp \omega_p) = \Theta(\pm p_0) \delta(p^2 - m^2) , \quad (4.149) \]

which lead to

\[ \Delta^{(\pm)}(x) = \mp i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \Theta(\pm p_0) \delta(p^2 - m^2) . \quad (4.150) \]

One further type of commutation function can be constructed by surrounding the two poles in opposite directions, using a contour \( C_1 \) shaped like the figure eight (see Fig. 4.4c):

\[ \Delta_1(x) = \Delta^{(+)}(x) - \Delta^{(-)}(x) . \quad (4.151) \]

The function \( \Delta_1(x) \) defined in this way\(^{15}\) has the three-dimensional Fourier representation

\[ \Delta_1(x) = \int_{C_1} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2} = -i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left( e^{-ip\cdot x} + e^{ip\cdot x} \right) \]

\[ = -i \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot x}}{\omega_p} \cos \omega_p x_0 . \quad (4.152) \]

An alternative representation follows from (4.150)

\[ \Delta_1(x) = -i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \delta(p^2 - m^2) . \quad (4.153) \]

\(^{15}\) Often the definition of \( \Delta_1(x) \) is given with an additional factor \( i \) which makes it a real-valued function.
We have encountered the function $i\Delta_1(x-y)$ in (4.121) where it was identified with the anticommutator of the field operators $\hat{\psi}(x)$ and $\hat{\psi}(y)$.

The commutation functions have the following properties under reflection and complex conjugation, which are easily verified from their integral representations:

\[
\begin{aligned}
\Delta^{(+)}(-x) &= -\Delta^{(-)}(x), \\
\Delta(-x) &= -\Delta(x), \\
\Delta_1(-x) &= \Delta_1(x), \\
\Delta^{(+)*}(x) &= \Delta^{(-)*}(x), \\
\Delta^*(x) &= \Delta(x) \quad \text{real}, \\
\Delta_1^*(x) &= -\Delta_1(x) \quad \text{imaginary}. \\
\end{aligned}
\] (4.154)

The Propagation Functions

All the functions introduced in the last section satisfy the homogeneous Klein–Gordon equation. This becomes immediately obvious from the four-dimensional Fourier representations (4.149), (4.150), and (4.153)) since $(p^2 - m^2)\delta(p^2 - m^2) = 0$. The Green's functions, on the other hand, can be constructed by solving the Fourier integral (4.147) with open integration contours that extend to infinity. Upon application of the operator $\Box + m^2$ the poles are canceled and we are left with an integral from $-\infty$ to $+\infty$. Since the integrand is the exponential function $\exp(-ip_0x_0)$ this leads to a delta function.

The most important representative of this class of functions is the Feynman propagator, which was introduced in Sect. 4.5:

\[
i\Delta_F(x-y) = \langle 0|T(\hat{\psi}(x)\hat{\psi}^\dagger(y))|0\rangle.
\] (4.155)

This function results from the integration

\[
\Delta_F(x) = \int_{C_F} \frac{d^4p}{(2\pi)^4} \frac{\exp(-ip\cdot x)}{p^2 - m^2} 
\] (4.156)

extended over the Feynman contour $C_F$ defined in Fig. 4.5a (see also Sect. 4.5).

Fig. 4.5. The integration contours that define the propagation functions.
The contour $C_F$ is open and extends to infinity. Since the integrand is a holomorphic function, the $p_0$ integration can be closed by adding a "half-circle at infinity". Here the sign of the time coordinate $x_0$ makes an important difference: the contour can be closed in the upper (lower) half plane provided that $x_0$ is negative (positive). The integration contour extended in this way encircles only one of the two poles. It can be deformed continuously without changing the value of the integral and is thus seen to be topologically equivalent either to the contour $C^{(+)}$ or to the contour $C^{(-)}$ from Fig. 4.4b. This obviously implies the relation

$$\Delta_F(x) = \Theta(x_0) \Delta^{(+)}(x) - \Theta(-x_0) \Delta^{(-)}(x). \tag{4.157}$$

Therefore the Feynman propagator corresponds to a sum of the positive-frequency Pauli–Jordan function for positive time arguments and of the negative-frequency Pauli–Jordan function for negative time arguments, both being combined with a negative relative sign. This construction of the Feynman propagator ensures the correct implementation of the condition of causality in the study of the propagation of perturbations. Here in the spirit of Stückelberg and Feynman antiparticles are treated as particles with negative frequency (energy) that move backward in time (see also the discussion in the volume Quantum Electrodynamics).

Alternative propagation functions can be defined through the open integration contours $C_A$ and $C_R$ in Fig. 4.5b, which pass both poles on the same side. Closing these contours can lead to the Pauli–Jordan function $\Delta(x)$ from Fig. 4.4a. If the time coordinate has the opposite sign, the integral vanishes since no poles are encircled. Thus we have

$$\Delta_R(x) = \Theta(x_0) \Delta(x), \tag{4.158a}$$

$$\Delta_A(x) = -\Theta(-x_0) \Delta(x). \tag{4.158b}$$

These are the retarded and the advanced Green’s function of the Klein–Gordon equation. $\Delta_R$ ($\Delta_A$) is nonvanishing only for a positive (negative) time argument. Looking at (4.158) we find that the Pauli–Jordan function $\Delta(x)$ can be written as the difference between the retarded and the advanced propagators:

$$\Delta(x) = \Delta_R(x) - \Delta_A(x). \tag{4.159}$$

This is quite natural since a comparison of Figs. 4.4a and 4.5b reveals that by pasting together the contours $C_R$ and $-C_A$ at infinity, we generate the integration contour $C$.

For completeness’ sake we mention the “anticausal” propagator (also known as the Dyson propagator) $\Delta_D(x)$, which encircles the poles in the opposite way compared to the Feynman propagator (see Fig. 4.5a):

$$\Delta_D(x) = \Theta(x_0) \Delta^{(-)}(x) - \Theta(-x_0) \Delta^{(+)} \tag{4.160}$$

Finally the principal-part propagator $\Delta(x)$ can be introduced, which is generated if the $p_0$ integration on the real axis runs through the poles at $p_0 = \pm \omega_p$ as sketched in Fig. 4.5c. The integration over the singularities is interpreted as a principal-part integral. This prescription can be interpreted as the arithmetic mean of two integrals along contours that pass the pole to the left or to the right. This implies
\[ \tilde{A}(x) = \frac{1}{2} (\Delta_R(x) + \Delta_A(x)) . \]  
(4.161)

According to (4.158) the principal-part propagator has a simple connection with the Pauli-Jordan function:
\[ \tilde{A}(x) = \frac{1}{2} \epsilon(x_0) \Delta(x) . \]  
(4.162)

A connection to the Feynman propagator is given through
\[ \Delta_F(x) = \tilde{A}(x) + \frac{1}{2} \Delta_1(x) . \]  
(4.163)

A further useful connection between the commutation and propagation functions is
\[ \Delta_1(x) = \Delta_F(x) - \Delta_D(x) . \]  
(4.164)

All the \( \Delta \) functions can be expressed in terms of two independent "basic" functions (since the integrand in the Fourier integral has two poles). A reasonable choice for these basic functions is \( \Delta(x) \) and \( \Delta_1(x) \). The last column of Table 4.1 on page 109 shows how the other \( \Delta \) functions can be obtained.

Obviously all the propagation functions contain a product of the function \( \Delta(x) \) with a unit step function in time \([\Theta(x_0) \text{ or } \frac{1}{2} \epsilon(x_0)]\). It is this step function that gives rise to the delta function when the Klein–Gordon operator is applied. For example the Klein–Gordon operator acts on the Feynman propagator as follows:

\[ (\Box + m^2)\Delta_F(x) = (\Box + m^2) \frac{1}{2} (\epsilon(x_0) \Delta(x) + \Delta_1(x)) \]
\[ = (\partial_0^2 - \nabla^2 + m^2) \frac{1}{2} \epsilon(x_0) \Delta(x) \]
\[ = (\partial_0 \delta(x_0)) \Delta(x) + 2 \delta(x_0) (\partial_0 \Delta(x)) + \frac{1}{2} \epsilon(x_0)(\Box + m^2) \Delta(x) . \]  
(4.165)

The last contribution vanishes since \( \Delta_1(x) \) and \( \Delta(x) \) solve the homogeneous Klein–Gordon equation. The first term containing the derivative of the delta function is equivalent to \( -\delta(x_0)(\partial_0 \Delta(x)) \). This is true since this expression has to be viewed as a distribution. Thus it makes sense only when it gets multiplied by a (sufficiently smooth) test function \( f(x_0) \) and integrated over \( x_0 \). Then an integration by parts leads to

\[ \int dx_0 (\partial_0 \delta(x_0)) \Delta(x) f(x_0) \]
\[ = - \int dx_0 \delta(x_0) (\partial_0 \Delta(x)) f - \int dx_0 \delta(x_0) \Delta(x) (\partial_0 f) \]
\[ = - \partial_0 \Delta(x) \big|_{x_0=0} f - \Delta(0, x) \partial_0 f \big|_{x_0=0} \]
\[ = - \partial_0 \Delta(x) \big|_{x_0=0} f . \]  
(4.166)

since \( \Delta(0, x) = 0 \) according to (4.104). The second boundary condition (see (4.105)), \( \partial_0 \Delta(x) \big|_{x_0=0} = -\epsilon^3(x) \), can be applied and (4.165) becomes
\[(\Box + m^2) \Delta_F(x) = -\delta^4(x), \quad (4.167)\]

as expected.

The functions \(\Delta(x)\) and \(\Delta_1(x)\), and as a consequence the whole family of \(\Delta\) functions, are expressed in terms of Fourier integrals that can be solved in closed form. Unfortunately the resulting expressions in coordinate space are quite unwieldy and are not very suitable for practical calculations. Usually it is much more convenient to work in momentum space. Nevertheless let us quote the resulting functions in coordinate space:\(^{16}\)

\[
\Delta(x) = -\frac{1}{2\pi} \epsilon(x_0) \delta(x^2) + \frac{m}{4\pi \sqrt{x^2}} \epsilon(x_0) \Theta(x^2) J_1(m\sqrt{x^2}), \quad (4.188a)
\]

\[
i\Delta_1(x) = \frac{m}{4\pi \sqrt{x^2}} \Theta(x^2) N_1(m\sqrt{x^2}) + \frac{m}{2\pi^2 \sqrt{-x^2}} \Theta(-x^2) K_1(m\sqrt{-x^2}). \quad (4.188b)
\]

These expressions contain the Bessel function \(J_1\), the Neumann function \(N_1\), and the MacDonald function \(K_1\) (which is a Hankel function of an imaginary argument), all of first order. For large values of the argument \(x\) the functions \(J_1(x)\) and \(N_1(x)\) are oscillatory, whereas \(K_1(x)\) falls off exponentially. From (4.168) we learn that the Pauli-Jordan function \(\Delta(x)\) vanishes identically for space-like separations \((x^2 < 0)\). In Sect. 4.4 this property was found to guarantee the condition of microcausality. In contrast to this, the function \(\Delta_1(x)\) and thus also the Feynman propagator \(\Delta_F(x)\) extend into the space-like region, dropping off on the scale of the Compton wavelength \(1/m\).

All the \(\Delta\) functions show singular behavior on the light cone. Taking into account the singularities of the Bessel functions at argument zero, we find the functional dependence in the vicinity of the light cone:

\[
\Delta(x) \simeq -\frac{1}{2\pi} \epsilon(x_0) \delta(x^2) + \frac{m^2}{8\pi} \epsilon(x_0) \Theta(x^2), \quad (4.169a)
\]

\[
i\Delta_1(x) \simeq -\frac{1}{2\pi^2 x^2} + \frac{m^2}{4\pi^2} \ln \frac{m\sqrt{|x^2|}}{2}. \quad (4.169b)
\]

Four different types of singularities are seen to arise at the light cone: \(\delta(x^2)\), \(\Theta(x^2)\), \(1/x^2\), and \(\ln|x^2|\).