(3.1) For boson operators satisfying
\[
\left[ \hat{a}_p, \hat{a}^\dagger_q \right] = \delta_{pq},
\]
show that
\[
\frac{1}{V} \sum_{pq} e^{i(p \cdot x - q \cdot y)} \left[ \hat{a}_p, \hat{a}^\dagger_q \right] = \delta^{(3)}(x - y),
\]
where \( V \) is the volume of space over which the system is defined. Repeat this for fermion commutation operators.

(3.2) Show that for the simple harmonic oscillator:
(a) \( [\hat{a}, (\hat{a}^\dagger)^n] = n!(\hat{a}^\dagger)^{n-1} \),
(b) \( \langle 0 | \hat{a}^n (\hat{a}^\dagger)^m | 0 \rangle = n! \delta_{n,m} \),
(c) \( \langle m | \hat{a}^\dagger | n \rangle = \sqrt{n + 1} \delta_{m,n+1} \),
(d) \( \langle m | \hat{a} | n \rangle = \sqrt{n} \delta_{m,n-1} \).

(5.6) Use the Lagrangian \( L = -\frac{mc^2}{\gamma} + qA \cdot v - qV \) for a free particle of charge \( q \) and mass \( m \) in an electromagnetic field to derive the Lorentz force, i.e. show
\[
\frac{d}{dt}(\gamma mv) = q(E + v \times B).
\]
[Hint: \( \nabla (a \cdot b) = (a \cdot \nabla)b + (b \cdot \nabla)a + a \times \text{curl} \ a + a \times \text{curl} \ b \).]

(5.7) Show that for the Lagrangian
\[
L = -\frac{mc^2}{\gamma} + qA \cdot v - qV,
\]
when \( v \ll c \) the momentum becomes \( p = mv + qA \) and the energy becomes
\[
E = mc^2 + \frac{1}{2m} (p - qA)^2 + qV.
\]
(16.4) (a) By taking the Fourier transform of the equation

$$(\nabla^2 + k^2)G_k(x) = \delta^{(3)}(x), \quad (16.45)$$

show that the momentum Green's function implied is

$$\tilde{G}_k(q) = \frac{1}{k^2 - q^2}. \quad (16.46)$$

Note that $\tilde{G}_k(q)$ is undefined when $q^2 = k^2$. This reflects the fact that there is an ambiguity in $\tilde{G}_k(q)$ in that we don't know if the wave is incoming, outgoing or a standing wave. This is sorted out with

an iε factor as shown in the next part.

(b) Take the Fourier transform of $G_\pm^k(x) = \frac{e^{i|x|\sigma}}{4\pi|x|}$ to show that, for outgoing waves,

$$\tilde{G}_\pm^k(q) = \frac{1}{k^2 - q^2 + i\epsilon}. \quad (16.47)$$

Hint: To ensure outgoing waves, you could include a damping factor $e^{-|x|}$ to damp out waves for $|x| \to \infty$. 