Physics 571 Lecture #13

1 Optimal Output Coupling

In our last lecture, we wrote the threshold gain as

$$g_t = -\frac{1}{2L} \ln(R_1 R_2) = -\frac{1}{2L} \ln[1 - (1 - R_1 R_2)].$$

(1)

The quantity in parentheses inside the argument of the logarithm is a small number. We can use the small-$x$ expansion of $\ln(1 - x) \approx -x$ to write the threshold gain as

$$g_t = \frac{1}{2L} (1 - R_1 R_2) + a,$$

(2)

where $a$ represents any absorption losses in the cavity.

Also in our last lecture we talked about the light intensity circulating inside the laser cavity shown in Fig. 1. We used the continuity equation in the uniform field approximation to write

$$\frac{1}{c} \frac{\partial}{\partial t} (I_+ + I_-) = g(\nu)(I_+ + I_-).$$

(3)

Let’s use the idea of gain saturation from a few lectures ago to study this equation a little further.

![Figure 1: A laser cavity with a gain medium. The intensity of light moving to the right is labeled $I_+$. The intensity of light moving to the left is labeled $I_-$. We can write rate equations for these quantities.](image)

We saw that the gain coefficient saturates at high values of the intensity. The gain coefficient was written as

$$g(\nu) = \frac{g_0(\nu_{21})}{1 + (\nu - \nu_{21})^2 / \Delta \nu_{21}^2 + I(\nu)/I_{sat}(\nu_{21})}.$$  

(4)

We will restrict our discussion to $\nu = \nu_{21}$ and simplify this a little to be

$$g = \frac{g_0}{1 + I/I_{sat}}.$$

(5)
For a laser in the steady state, the intensity grows until the gains equal the losses. When that happens, the intensity is as high as it can be. Therefore, we can say that the steady state gain is always the threshold gain. If it wasn’t, the intensity would keep growing and therefore not be in a steady state or uniform field condition. Furthermore, we can identify the intensity in the laser cavity as \( I = I_+ + I_- \). Therefore, we can write Eq. 5 as

\[
g_t = \frac{g_0}{1 + (I_+ + I_-)/I_{\text{sat}}} \Rightarrow I_+ + I_- = I_{\text{sat}} \left( \frac{g_0}{g_t} - 1 \right) .
\] (6)

Now things get interesting. Let’s suppose that \( R_1 = 1 \) and \( I_+ \approx I_- \). At mirror 2, the light can be transmitted, reflected, or perhaps scattered by imperfections in the mirror. If we consider the conservation of energy \( (T + R + S = 1) \), the transmission of mirror 2 is \( T = 1 - R - S \ll 1 \). For what follows, we will neglect scattering, but you can see how it should be put into the equations. The output intensity is

\[
I_{\text{out}} = TI_+ = \frac{T}{2} I_{\text{sat}} \left( \frac{g_0}{g_t} - 1 \right) .
\] (7)

The output power is plotted in Fig. 2

![Figure 2: Laser intensity coupled out of a two-mirror cavity as a function of the transmission of the output coupler (mirror 2 in Fig. 1). For a finite absorption loss \( a \), there is an optimal non-zero value of the mirror transmission. Notice that the maximum output intensity approaches \( g_0 I_{\text{sat}} \) as \( a \to 0 \).]

Using our definition of the threshold gain in Eq. 2, we can write

\[
I_{\text{out}} = \frac{T}{2} I_{\text{sat}} \left( \frac{2Lg_0}{T + 2La} - 1 \right) .
\] (8)

Finding the optimum output coupling now is simply a calculus problem. Setting the derivative equal to zero and solving for \( T \), we find

\[
T_{\text{opt}} = \frac{2L}{-a \pm \sqrt{a}g_0} .
\] (9)

We take the positive sign so that \( T \) can be positive. If we plug this back into our expression for the threshold gain, we find that

\[
g_{t,\text{opt}} = \sqrt{a}g_0 .
\] (10)
2  A tiny rehash

OK. Let’s review a little. We derived the threshold gain, \( g_t = \frac{1 - R_1 R_2}{2L} \), for a two-mirror linear cavity. We discovered that in the steady state, the gain must always be equal to the threshold gain(!). We used full expression for how the gain depends on the intensity inside the laser cavity (Eq. 5), set this equal to \( g_t \), solved for \( I_+ \), and calculated the output intensity. Obviously, the optimum output intensity depends on the gain and loss inside the cavity, and we used a little calculus to find the optimum output coupling, \( T_{opt} \). We can put this optimum value back into Eq. 7 to see what the “best” output intensity will be. A little algebra shows that it is

\[
I_{out, opt} = I_{sat} \left[ L g_0 \left( 1 + \frac{a}{g_0} - 2 \sqrt{\frac{a}{g_0}} \right) \right].
\]

(11)

The optimum output power is maximized when the absorption is small (of course).

But Eq. 11 looks a little puzzling at first. If the steady state gain is always the threshold gain, how do I get any power out of the laser at all? The answer is in \( g_0 \). Remember that the small signal gain is proportional to the (small signal) population inversion,

\[
g_0 = \sigma(\nu)(N_2 - N_1).
\]

(12)

The cross section \( \sigma \) is a materials property, but the (small signal) population inversion depends on pumping. For a three-level laser in steady state, the population inversion is

\[
N_2 - N_1 = \frac{P - \Gamma_{21} N_T}{P + \Gamma_{21} N_T}.
\]

(13)

A plot of the scaled population inversion \( \tilde{N} \equiv (N_2 - N_1)/N_T \) vs. the scaled power \( \tilde{P} = P/\Gamma_{21} \) is shown in Fig. 3. When the absorption \( a \) is small in Eq. 11, the output power depends linearly on the small signal population inversion.

![Graph of \( \tilde{N} \) vs. \( \tilde{P} \).](image)

Figure 3: A graph of \( \tilde{N} \) vs. \( \tilde{P} \).

In many lasers, the output power scales more or less linearly with the pump power. One example is shown in Fig. 4. This is the output power from an injection-seeded cw sapphire laser (a four level laser system). The important feature is that above threshold, the laser output power is essentially linear with the pump power. The population inversion is not approaching saturation, and continuing to turn up the pump source will continue to increase the output power (assuming we don’t break something in the process). Changing the output coupler changes the \( y = 0 \) intercept of the line (called the threshold power) and the slope of the line (called the efficiency slope).
Figure 4: Output power vs. pump power for an injection-seeded Ti:sapphire laser. The black and white circles show the output power with and without a cw injection laser. [E. A. Cummings, M. S. Hicken, and S. D Bergeson, Applied Optics 36, 7583-7587 (2002)]