Lesson 5 – Representing Fields Geometrically
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5.0 Introduction

In Lesson 3 we introduced the idea of electric and magnetic field lines and field contours. We learned that for a single point charge, the threads could be aligned to produce electric field lines and the stubs aligned to produce magnetic field lines. But what happens when we have many source particles?

For a single source particle, the force on a field particle in thread model was given by the equation:

\[ \vec{F} = \frac{e}{\varepsilon_0} q_f \nu \vec{\ell} \]

where \( q_f \) is the charge of the field particle in coulombs (C)
\( \nu \) is the density of threads in number/m³.
\( \vec{\ell} \) is the vector thread length in meters.

This implies that for a single source particle,

\[ \vec{E}_i = \frac{e}{\varepsilon_0 \nu} \vec{\ell}_i = \frac{e}{\varepsilon_0 \Delta V} \vec{\ell}_i \]

where the \( i \) subscripts specify the values are for the \( i \)th source particle, \( N \) is the number of threads in some small volume \( \Delta V \). Then the total electric field at a point is given by the sum of this expression over all the threads in the region.

\[ \vec{E} = \frac{e}{\varepsilon_0 \Delta V} \sum_{i=1}^{\text{# of sources}} N_i \vec{\ell}_i. \]

Instead of summing over the sources, we could accomplish the same thing be a summation over all the threads in the volume.

(5.1) \[ \vec{E} = \frac{e}{\varepsilon_0 \Delta V} \sum_{j=1}^{\text{# of threads}} \vec{\ell}_j. \]

What this relation tells us is that we can find the electric field in a given region by adding all the threads as vectors in a small volume, dividing by the volume, and multiplying by constants. However, for many sources charges, this can be a rather difficult process.

We also learned in Lesson 2 that we can align the threads of a single source to create electric field lines and align the stubs of that source to create magnetic field lines. It turns out
that another way of dealing with many sources is to take the sum over all threads in Eq. (5.1) above and replace the sum with a single effective thread:

\[
\vec{E} = \frac{e}{\varepsilon_0} \frac{1}{\Delta V} \vec{L}.
\]

Then we can align the \( \vec{L} \) vectors from different small volume elements to create a new set of electric field lines. So instead of having many overlapping sets of field lines, we are left with one single set of field lines. In a similar way, we could create a set of magnetic field lines as well. As you might guess, it can become difficult to draw field lines for complicated arrangements of source charges.

In this lesson we will learn how to sketch the electric field lines and contours of a few stationary charges and the magnetic field lines and contours of a few parallel, current-carrying wires. You will discover that we can understand many important details about electromagnetic interactions through the geometrical analysis of field lines and contours. In later chapters, we will look at some systematic properties of field lines and field contours to derive Maxwell’s Equations, the fundamental equations of modern electromagnetic theory.

We found that one way to form field lines was to start at one point, find the field, draw a very short line segment in the direction of the field, find the field at this new point, and continue in this same fashion. We can follow a similar process for the fields of multiple sources. Although these “bulk field lines” lose a bit of the information contained in threads and stubs, they prove much easier to use for many problems involving extended charge distributions. The following table summarizes some of the major differences between threads and field lines.

<table>
<thead>
<tr>
<th>threads</th>
<th>(bulk) electric field lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>very small, move radially outward from charges</td>
<td>start on positive charges and end on negative charges – or go off to infinity</td>
</tr>
<tr>
<td>move at the speed of light</td>
<td>changes “move” at the speed of light</td>
</tr>
<tr>
<td>go in straight lines</td>
<td>bend around, following the direction of the force</td>
</tr>
<tr>
<td>can not be destroyed, but their effects can be cancelled by other threads</td>
<td>do not exist in regions where there is no net force</td>
</tr>
<tr>
<td>magnetic fields result from motion correction to threads</td>
<td>magnetic fields result from 1) currents (unrelated to electric fields) or from 2) changing electric fields – but we won’t see the second case until later.</td>
</tr>
</tbody>
</table>

5.1 The Electric Field Lines of a Stationary Point Charge

We know that the electric field lines of a stationary point charge are straight lines directed radially outward from a positive charge or radially inward toward a negative charge. We also know that the larger the charge, the more field lines it has. Two typical arrangements of
field lines are shown in Figs. 5.1 and 5.2. (The units of charge in these figures are arbitrary.) We can arrive at these conclusions in two different ways.
First, let’s consider the threads of a point charge at rest. We know all the threads point radially outward for a positive charge and radially inward for a negative charge. The total number of threads emitted per unit time by a point charge is proportional to the magnitude of the charge. The electric field lines, obtained by aligning these threads, are therefore radial, uniformly distributed in space, and proportional to the charge.

The second method of determining the field lines of a point charge is to construct the field lines point by point as described in Sect. 3.4. We can begin with an arbitrary point near the charge and draw a short line segment from that point in the direction of the field. From the end of that segment, we draw a new segment in the new direction of the field, etc. In the case of a point charge, the field is always radially outward. (As you may recall, an easy way to find the direction of the electric field at a point is to determine the direction of a force on a small positive test charge located at that point.) Hence, each field line must be radially outward. We also know from Sect. 3.4 that the number of field lines per unit area must be proportional to the field. If the charge is larger, the field is proportionally larger from Coulomb’s law, and the number of total field lines must also be proportionally larger. The field lines must be symmetrically arranged in space, as we argued above. We again conclude that the electric field lines must be radial, uniformly distributed in space, and proportional to the charge.

Let us then summarize these rules:

Rules for the Electric Field Lines of Point Charges

1. The magnitude of an electric field is determined by the number of electric field lines per unit area passing through a section of a perpendicular surface.
2. Electric field lines point radially outward from positive charges and radially inward toward negative charges.
3. The number of field lines coming from or going into a charge is proportional to the size of the charge.
4. Field lines are uniformly distributed; that is, the field lines are not closer together in some directions than in others.

Note that the total number of field lines remains fixed. All the field lines start or stop on charges. No field lines spring up out of empty space or disappear into empty space. We also know that the perpendicular surfaces for these field lines are just spheres. The surface area of a sphere is \( A = 4\pi r^2 \). By applying these facts with the first rule, we may deduce that

\[
E \propto \frac{N}{4\pi r^2}
\]

where
- \( \propto \) means “is proportional to.”
- \( N \) is the total number of field lines coming from (or going toward) a charge.
- \( r \) is the distance from a charge.
Since we know that the total number of field lines is, in turn, proportional to the charge of the source particle, we can deduce that:

$$E \propto \frac{q}{4\pi r^2}$$

We see that, with a suitable choice of the constant of proportionality, this expression is completely equivalent to Coulomb’s law:

$$E = \frac{q}{4\pi \varepsilon_0 r^2}$$

What this means is that the same information that is algebraically expressed by Coulomb’s law is geometrically expressed by correctly drawn electric field lines. This is an important point that sometimes is hard to grasp since we are used to dealing with algebraic models and equations. To put it in different words, Fig. 5.1 and 5.2 are statements of Coulomb’s law.

In theory, we require the total number of field lines to be infinite; however, since we want to use drawings of field lines, we need to limit our figures to a few select field lines. In this book, I arbitrarily choose to draw four field lines per unit of charge. Note that this convention was used in Figs. 5.1 and 5.2.

Think About It

Look over Figs. 5.1 and 5.2 and satisfy yourself that the figures obey all the rules for the field lines of point charges.

Although the rules are simple enough to interpret, trying to draw three-dimensional objects accurately in two dimensions can become rather complicated. Most illustrations in textbooks make no attempt to depict three-dimensional fields. To simplify our drawings and our interpretation of the drawings, we usually reduce everything to two dimensions. Just keep in mind that just as a flat map of the earth’s surface has distortions, a flat map of electric field lines has distortions and inaccuracies. Again, let us use the convention that we draw four field lines per unit of charge. Figs. 5.3 and 5.4 are the new "flat" versions of Figs. 5.1 and 5.2.
Figure 5.3. The “flat” version of Fig. 5.1

Figure 5.4. The “flat” version of Fig. 5.2
Think About It

Use some clay and pipe stem cleaners (or something similar) to construct the field lines for charges of +1.5 and −2 units. Draw "flat" representations of the field lines for the same cases.

Things to remember:
• Electric field lines go radially outward from positive charges and radially inward toward negative charges.
• The field is proportional to the number of field lines per unit area (taken on a section of a perpendicular surface).
• The field lines of a point charge are uniformly distributed at all angles.
• Field lines are a geometrical way of expressing the physics of Coulomb’s law.
• We usually use “flat” drawings of field lines, but we need to remember that this is just a cartoon version of the real, three-dimensional world.

5.2 Electric Field Contours of Point Charges

Whenever we draw the electric field lines of a point charge according to the rules of the previous section, we incorporate Coulomb's law into our drawing. The field contour is another geometrical construction that is also useful in working with fields. A field contour is, as we learned in Sect. 3.4, a set of surfaces each of which is perpendicular to the field lines. The surfaces also have to be spaced in such a way that the field strength is inversely proportional to the spacing between the surfaces.

For point charges, it is easy to find the perpendicular surfaces; they are just spheres centered on the charge. The field lines are naturally close together where the field is stronger; however, we have no constraints at all on where we can construct spheres around a charge. So how do we select spheres that are properly spaced? This process is just a bit tricky, so pay careful attention! We will use some numbers to make things more concrete, but if you have a mathematical inclination, you may want to reproduce the argument with symbols.

Example 5.1. Constructing a Field Contour.

First, let’s choose the charge to be \( q = 4\pi \times 8.85 \times 10^{-12} \) C. This rather awkward looking charge numerically cancels out the \( 4\pi\varepsilon_0 \) in the denominator. Then, as long we put \( r \) in meters, we have \( E = 1/r^2 \) with \( E \) in V/m. Furthermore, the separation between surfaces in the contour needs to be proportional to \( 1/E \) so that the field is twice as strong when the surfaces are twice as close together. The separation between contours is then \( \Delta r \propto 1/E \propto r^2 \). For this to be a meaningful relationship, we want \( \Delta r \ll r \) so \( r^2 \) doesn’t vary a lot between adjacent surfaces of the contour.
So, let’s take $\Delta r$ to be something small, like $1 \text{ mm}$. Then, letting $r = 2 \text{ m}$ and $\alpha$ be a constant, we have:

\[
\Delta r = \frac{\alpha}{E}
\]

\[
\alpha = E\Delta r = \frac{1 V}{4 m} \times 0.001m = 0.00025 V
\]

Now that we know $\alpha$, we can find the separation between spheres at any distance. For example, if $r = 1 \text{ m}$, then we have:

\[
\Delta r = \frac{\alpha}{E} = \frac{0.00025 V}{1V/m} = 0.00025m = 0.25 \text{ mm}
\]

As expected, the field is four times larger, so the spacing between spheres is four times smaller.

You may have noticed that the constant, $\alpha$, has units of volts. If we look at Eqs. (3.6), we see that it’s no accident for $E\Delta r = -\Delta V$. (The minus sign just tells us that if the charge is positive, the potential gets smaller when we move to larger $r$.) Since the same $\alpha$ applies to the separation between all pairs of spherical surfaces in the field contour, we see that all we have to do to get a good field contour is to choose a set of spheres that are $0.00025 \text{ V}$ apart in their electric potentials.

Mathematically, that’s just fine, but we don’t want to draw thousands of spheres when we make graphical representations of the field contour. We need to choose a few typical surfaces for our drawing. But that’s really quite easy; we only have to choose every 1000 spheres, for example. In that case, we pick spheres that are $1000 \times 0.00025 \text{ V} = 0.25 \text{ V}$ apart in electric potential. – So, let’s do it! First, we recall that the expression for the electric potential of a point charge is

\[
V = \frac{q}{4\pi\varepsilon_0 r} = \frac{1}{r} \text{ in SI units with our choice of } q.
\]

At infinity $V = 0$, so let’s work from there and find the values of $r$ for $V = 0.25 \text{ V}$, $0.50 \text{ V}$, etc.

\[
V = 0.25 \text{ V}, \quad r = 4.00 \text{ m}
\]

\[
V = 0.50 \text{ V}, \quad r = 2.00 \text{ m}
\]

\[
V = 0.75 \text{ V}, \quad r = 1.33 \text{ m}
\]

\[
V = 1.00 \text{ V}, \quad r = 1.00 \text{ m}
\]

\[
V = 1.25 \text{ V}, \quad r = 0.80 \text{ m}
\]
In case you’re wondering, you won’t have to do this sort of calculation often. This exercise does, however, point out a few important things:

1. The correct spacing between perpendicular surfaces in electric field contours does not “just happen” the way the correct spacing between field lines does.
2. Each perpendicular surface was at a fixed value of electric potential. For this reason the surfaces are also called “equipotential surfaces” for static electric fields. All this really says is that when you move perpendicularly to a force, as when you move horizontally in the earth’s gravitational field, you don’t change your potential energy.
3. To form a field contour, we select surfaces whose electric potentials are separated by a fixed number of volts. It doesn’t matter if the electric potentials of the surfaces are 1 V apart or 10V apart, as long the voltage difference between adjacent surfaces always has the same value.

Things to remember:
- For point charges, perpendicular surfaces are spheres with the charge at the center.
- Perpendicular surfaces are also equipotential surfaces for any static distribution of charge.
- Adjacent surfaces in a field contour for static electric charges are separated by a fixed voltage.

5.3 The Magnetic Field of a Current-Carrying Wire

When physicists first began to recognize the connections between electricity and magnetism, they thought of the fundamental source of magnetic field to be an infinitely-long, very thin, current-carrying wire. The magnetic field of such a wire is significantly simpler than the magnetic field of a single moving charge, so we will follow the historical development and consider long, current-carrying wires to be the fundamental source of magnetic field. The only disadvantage in thinking of such a field as elementary is that the simple relationship between electricity and magnetism is lost for a time.

In Sect. 2.9 we found that the force on a charge \( q_f \) moving with velocity \( v_f = \beta_f c \) parallel to a wire with electrons of density \( \lambda \) and drift speed \( v_s = \beta_s c \) was

\[
F(r) = \frac{q_f \beta_f \lambda \beta_s}{2\pi \epsilon_0 r}.
\]

We also know the magnetic field lines form loops around the wire in the direction given by the right-hand rule. This means that the velocity of the field particles is at 90° to the magnetic field and the Lorentz Force is just:
We usually know the current in the wire but seldom know the drift speed and charge densities, so let’s rewrite this equation in terms of current. Since $\lambda$ is the charge per unit length in the wire and $v_s$ is the average speed of the electrons in the wire, the current is just $i = \lambda v_s = \lambda \beta c$. Therefore:

$$B(r) = \frac{1}{c^2} \frac{\lambda \beta_c}{2\pi \varepsilon_0 r} = \frac{1}{c^2} \frac{i}{2\pi \varepsilon_0 r}$$

For convenience, we define a new constant, the “permeability of free space” or “mu-naught” as

$$\mu_0 = \frac{1}{\varepsilon_0 c^2} = 4\pi \times 10^{-7} \text{ Tm/} \text{A}$$

This gives us:

The Magnetic Field of a Long Wire

$$B(r) = \frac{\mu_0 i}{2\pi r}$$

where

- $B(r)$ is the magnitude of the magnetic field at a distance $r$ from the wire. It is measured in tesla (T).
- $i$ is the current in amperes (A).
- $\mu_0$ is the permeability of free space = 4$\pi$ $\times$ 10$^{-7}$ Tm/ A.

The direction of the field is given by the right-hand rule: Grasp the wire with your right hand such that your thumb points in the direction of the current. The magnetic field lines go around the wire in the direction of your fingers.

We now want to construct magnetic field lines for a current-carrying wire so that they can be interpreted in the same way as electric field lines. That is, we require the following:
Rules for Magnetic Field Lines

1. Magnetic field lines always point in the direction of the magnetic field.
2. The number of magnetic field lines per unit area gives the strength of the magnetic field.

Whereas the electric field lines naturally carry the $q/r^2$ dependence of Coulomb's law, the magnetic field lines do not naturally have the $i/r$ dependence we require. We are forced to draw the field lines at appropriate locations along the length of the wire and appropriate values of $r$ so that the number of field lines per unit area is proportional to the magnetic field. In practice, this is difficult to do; however, we try to draw the field lines closer together where the field is stronger.

A three dimensional drawing of the magnetic field lines of a wire is shown in Fig. 5.5.

![Figure 5.5. Magnetic field lines around a long wire.](image)

Try the right hand rule to be sure you understand how to determine the direction of these magnetic field lines.

Things to remember:
- The magnetic field of a long, current-carrying wire is $B(r) = \mu_0 i / 2\pi r$.
- The direction of magnetic field lines around a wire is given by the right-hand rule.
- Magnetic field lines are similar to electric field lines in that they are close together where the field is stronger and the field is tangent to the field lines.
- The magnetic field lines have to be drawn at correct locations, much as electric field contours.
5.4 Magnetic Field Contours

We can construct magnetic field contours in the same way we constructed electric field contours. First, we need to find the perpendicular surfaces. For the magnetic field of a long wire, the perpendicular surfaces are flat planes – or really half-planes since planes have infinite extent in all directions – with the wire forming the boundary of each half-plane. At a fixed distance from the wire, the magnetic field must be the same at all angles. Hence, the perpendicular surfaces must be arranged symmetrically about the wire. This is illustrated in Fig. 5.6.

![Figure 5.6. Magnetic field lines (orange) and contour (dotted red) of a long wire.](image)

It is helpful to assign directions to the perpendicular surfaces. We simply let the direction be given by the direction of the magnetic field at any given point on the surface.

Note that for the electric fields of stationary charges, the perpendicular surfaces have the physical interpretation of being equipotential surfaces. In the magnetic field case, there is no such thing as an equipotential surface, and the perpendicular surfaces have no particular physical significance.

With the electric field, we went to considerable effort to determine where we should draw the surfaces of the contour. You may have noticed that for the magnetic field we really had no freedom how to draw the surfaces except, perhaps, to choose the number of surfaces we would draw. To be a valid field contour, we know the surfaces must be closer together where the field is stronger. Visually, the surfaces of our drawing seem to meet this criterion, but let’s check out the math. First, follow a path around a magnetic field line located at a distance $r$ from the wire. As we follow the field line, we pass through a number of field contours, say $N$ contours. (In Fig. 5.6 $N = 3$.) The magnetic field is proportional to the number of contours per unit length, $N / 2\pi r$. Since $N$ is always the same at any value of $r$, we see that the expected $1/r$ dependence of the magnetic field is satisfied. We also know that $B$ must be directly proportional to the current.
Therefore, we must let the number of contours we draw be proportional to the current in the wire. In this book, I arbitrarily chose to draw three contours per unit of current.

Figs. 5.7 – 5.9 are two-dimensional representations of the magnetic field lines and contours of three different wires carrying current into the screen or out of the screen.

![Diagram of magnetic field lines and contours](image)

Figure 5.7. The magnetic field lines and contour of one unit of current going into the screen.

![Diagram of magnetic field lines and contours](image)

Figure 5.8. The magnetic field lines and contour of two units of current going into the screen.
Figure 5.9. The magnetic field lines and contour of two units of current coming out of the screen.

Things to remember:
- The $I/r$ dependence of the magnetic field of a wire is naturally built into the magnetic field contour of the wire.
- The direction of a perpendicular surface is defined as the direction of the magnetic field on the surface.

5.5 Comparing Electric and Magnetic Fields

Although we treat the electric field of a point charge and the magnetic field of a current-carrying wire in much the same way, we can see that there are several fundamental differences between them. As we continue to study these fields, we will see they are typical examples of two distinct classes of fields. For now, let's summarize some of the facts we know:

<table>
<thead>
<tr>
<th>Electric Field</th>
<th>Magnetic Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point source</td>
<td>Current source</td>
</tr>
<tr>
<td>Field lines begin and end on charges</td>
<td>Field lines form closed loops</td>
</tr>
<tr>
<td>Field lines extend to infinity</td>
<td>Field lines remain localized</td>
</tr>
<tr>
<td>$q/r^2$ dependence is in the field lines</td>
<td>$i/r$ dependence is in the field contours</td>
</tr>
<tr>
<td>Perpendicular surfaces are closed; that is, the surfaces enclose the charge</td>
<td>Perpendicular surfaces are open</td>
</tr>
<tr>
<td>Perpendicular surfaces remain localized</td>
<td>Perpendicular surfaces extend to infinity</td>
</tr>
<tr>
<td>Perpendicular surfaces are equipotential surfaces</td>
<td>Perpendicular surfaces have no physical interpretation</td>
</tr>
</tbody>
</table>
5.6 Electric Field Lines of Multiple Point Charges

In the previous sections we learned about the field line model and discussed the electric and magnetic field lines of elementary field sources. Of course, we are most often interested in the fields of more complicated systems than point charges and infinitely-long, current-carrying wires. What we need to do is generalize the results of the previous sections to handle more complicated systems. The problem with this is that there is usually no unique way to generalize from simple to complex. The approach we take is typical: 1) we decide what characteristics of field lines we think should apply to all systems, 2) we make an educated guess how to generate field lines for complicated systems, and 3) test our guess to see if it satisfies the characteristics we felt were important.

There are three criteria that are essential in characterizing electric field lines for the general case:

General Rules for Field Lines

1. Field lines always point in the direction of the field vectors.

2. The strength of the field is proportional to the number of field lines per unit area. (The field is stronger where field lines are closer together.)

3. The general rules for field lines must reduce to the rules for elementary sources for any system which is identical to an elementary source.

We extensively used the first two rules in the previous module so they should need no further explanation. The third rule is one of the foundations of theoretical physics. In quantum mechanics this idea is known as the "Correspondence Principle" and is one of the pillars of the development of modern physics. The idea is simple: if an old theory works, any new theory, to be acceptable, has to reduce to the old theory under proper circumstances. Relativistic mechanics has to give the same results as classical mechanics whenever the velocity of objects is very small compared to the speed of light. Quantum mechanics has to give the same results as classical mechanics whenever systems are "large." In our case, anything that is similar to a point charge has to have the field lines of a point charge as described in Sect. 5.1.

Think About It

Consider a system of three point charges located in the vicinity of each other. The magnitudes of the charges in arbitrary units are +3, −1, and +1. In what regions of space must the field lines resemble the field lines of point particles?
The near field: We know that the field lines of a point charge depend on the distance from the charge as $1/r^2$. This means that the field near each particle becomes very strong as we approach the particle. In fact, the field becomes so much stronger than the fields of the other particles that we can completely ignore the other particles. Hence, very near each point charge, the fields must be the same as the field of a single point particle. In theory, of course, the field is never identical to that of a point particle. In practice, however, the small contributions from the other fields are unimportant. The field near particles is called the "near field."

The far field: If we are very far away from a collection of particles, we can no longer distinguish one particle from another. The entire collection looks the same as a single point particle whose charge is the sum of all the individual charges. Hence, If we are much farther away from some charges than the maximum distance of separation between any two individual particles, the electric field must be the same as the electric field of a single point charge. This is called the "far field."

Knowing that the near field and the far field are like the fields of point charges can be useful; however, it's also important to know what the intermediate field is like. Since we require the field lines to point in the direction of the field vectors at every point in space, it is reasonable to follow the following prescription:

Tracing Electric Field Lines

1. Construct near field lines as field lines of point particles.


3. Continue the intermediate field lines to the far field guided by the knowledge that far field lines are like the field lines of point charges.

Things to remember:

• The near field and far field are similar to the fields of point charges.
• The direction of a field line at any point in space is the direction of the force on a small test charge.

5.7 Drawing Fields of Multiple Point Sources

Now that we’ve decided how to draw field lines for multiple sources, let’s actually do it. First, we take the case of two charges of equal magnitude and opposite sign. In this case we see that the near field lines are those of a point positive and negative charge. Although we may carefully calculate the direction of the electric field at closely spaced points along a field line (this was done to generate the figures in this chapter), we can estimate the direction of the electric field just by asking ourselves the direction of force experienced by a small, positive test
charge. We must be careful, though, not to follow the trajectory of a test charge, as that will be influenced by the charge's velocity as well as its position! We recognize that as we move away from the positive charge, the attractive force of the negative charge begins to play a greater role. Thus each field line from the positive charge is bent around until it comes to the negative charge.

![Figure 5.10. The field lines of two opposite charges in three dimensions.](image1)

![Figure 5.11. Flat representation of the field lines of two opposite charges.](image2)

As in the case of single charges, it is often confusing and difficult to draw field lines in three dimensions. Consequently, we often rely on "flat" representations of the field lines, as in Fig. 5.11.

In either case, notice that there is no unique way to draw the pictures. In Fig. 5.11, for example, we could have had field lines coming from the charges in horizontal and vertical directions. Since we usually use field lines to visualize the field, we are more concerned about the lines being qualitatively correct than we are about every detail of how to draw them.

Now, let us look at the case of equal charges. Note again that the behavior of the near field lines is as expected. This time, as we move away from either charge, the second charge provides a repulsive force to a small, positive test charge, so the lines go off to infinity. As we draw the field lines of two equal particles in a flat representation, we see that the same general features are preserved.
Figure 5.12. The field lines two equal charges in three dimensions.

Figure 5.13. Flat representation of the field lines of two equal charges.

Think About It

Do the far field lines in Figs. 5.10 through 5.13 agree with what you would expect? Are there any distortions in the flat representations which might be misleading? In Fig. 5.13 we could have begun drawing a field line from the left-hand particle directed to the right. What would happen to this field line? We often omit the field lines going directly from one particle toward the other.
Now, let’s look at the electric fields of two particles which have charges that are neither equal nor opposite each other. Note that in the case illustrated in Fig. 5.14, each of the four electric field lines coming out from the +1 charge is "trapped" by the −2 charge. Also note that the −2 charge has four additional lines that come in from infinity. A similar drawing is shown in Fig. 5.15 for charges that have the same sign but which are unequal in magnitude.

![Figure 5.14. Electric field lines of charges +1 and −2.](image)

![Figure 5.15 Electric field lines of charges +1 and +3.](image)

Things to remember:
- Be sure that you understand each of the drawings in this section and that you can make correct drawings of electric fields for two or three point charges.
- Correct drawings must have the correct near field behavior and the correct far field behavior.
- Correct drawings must have field lines starting on positive charges and ending on negative charges.
- Correct drawings must have the number of field lines be proportional to the charge.
5.8 Symmetric Distributions of Charge

One of the most powerful tools in theoretical physics is symmetry. Symmetry principles have important applications in areas as diverse as the structure of solids and predicting the existence of new elementary particles. And symmetry has many important applications in electricity and magnetism as well. In this section we will use symmetry arguments to predict the nature of electric field lines from a handful of highly symmetric but important configurations of charge.

Let us first consider a uniformly charged sphere; that is, the sphere is made up of a very large number of small positive point charges. Let us place a small, positive test charge on the surface of this sphere. What direction is the force on this charge? It should be intuitive that the direction of the force is directly outward from the center of the sphere. Why is that so? Because there is nothing special about one side or the other. In Fig. 5.16, there is nothing at all that distinguishes the direction to the right from the direction to the left. Since neither right nor left is special, the only option is for the electric field to go neither right nor left. Hence the field is directed radially outward from the sphere.

![Figure 5.16. The electric field on the surface of a uniformly charged sphere.](image)

Similarly, we can ask if there are more field lines coming from one part of the sphere than another. The answer must be no because each part of the sphere's surface is equivalent to each other part. The number of field lines per unit area must be the same at every point on the surface of the sphere.

We say that an object or a function has "radial spherical symmetry" or just "radial symmetry" if there is no dependence on the angles $\theta$ and $\phi$ in spherical coordinates. A sphere has radial symmetry, and a spherical shell (with a hollow center) also has radial symmetry. A sphere with charge only on part of the surface, however, does not have radial symmetry. Clearly any
A charged object with radial symmetry has electric field lines which are directed radially outward from the center of the sphere or inward toward the center of the sphere.

**Think About It**

By drawing an analogy between the electric field of a point charge and the electric field outside a uniformly charged spherical shell, can you determine the electric field outside such a shell?

We say that something has "radial cylindrical symmetry" or just "radial symmetry" if, in cylindrical coordinates, it has dependence only on \( r \) and not on \( \theta \) or \( z \). An infinitely long, uniformly charged rod has radial symmetry. A finite, uniformly charged rod does not have radial symmetry since there is \( z \) dependence to the charge distribution. In fact, far from a finite charged rod, the field must become that of a point charge.

**Think About It**

Describe the electric field lines near the end of a finite, uniformly-charged, cylindrical rod.

Using exactly the same argument as we did for radial spherical symmetry, we can demonstrate that the electric field lines of any object with radial cylindrical symmetry must be directed radially outward from the axis of the cylinder.

One last type of symmetry that is very important is that of an infinite plane which is uniform in both directions. We term this "planar symmetry." To have such symmetry, a charge distribution must be uniform over an entire plane of infinite extent. In such a case, we see that symmetry demands that the electric field must be perpendicular to the plane everywhere. If we think of the screen as forming part of the infinite plane of charge, we see that there can be no component of the electric field to the right or left, up or down, because there is nothing to distinguish one direction from any other direction.

**Things to remember:**

- In this course, there are three important symmetries we need to consider: radial spherical, radial cylindrical, and planar.
- By invoking symmetry, we can simplify many problems.
5.9 The Magnetic Field of Two Parallel Wires

We took as our elementary source of magnetic fields an infinitely-long, current-carrying wire. The next level of complexity is to have two such wires parallel to each other. In this case, we can take an approach very similar to that which we took in the electric field case.

In regions which are much nearer one wire than the other, the fields will be dominated by the nearer wire. Hence, the near field will be approximately the field of an individual wire. In regions which are much farther away from either wire than the separation distance of the wires, the net field will resemble the field of a single wire carrying the total current (taking the direction of the current into consideration, of course) of both wires. Thus, the far field will also be approximately the same as the magnetic field of a single wire.

Thus in the near field and the far field, the field lines will be circles. In the intermediate region, the field lines will no longer be circular. Close to the near field region, the lines will be closer together on the side where the field is stronger (look at the fields from the two wires individually to determine this) and farther away on the side where the field is weaker. After that, we blend the field lines together to form one set of field lines in the far field.

Now let us look at a few examples. In the first case we have two wires carrying the same current in the same direction. As with a single wire, we only need to consider the magnetic field in a plane perpendicular to the wires. Note that in the region between the wires the magnetic field from each wire is in the opposite direction, so the field cancels. This means that the field lines are farther apart between the wires. On the other hand, on the outside the two wires, the fields are in approximately the same direction, so they add. Notice the characteristics of the near field and the far field.

Figure 5.17. The magnetic field of two wires carrying equal currents out of the screen.
If we have two wires with unequal currents traveling in the same direction, as in Fig. 5.18, note that a similar situation holds; however, the smaller field makes a relatively small perturbation to the larger field.

In the case that the currents of the two wires are in opposite direction, the field between the two wires adds, whereas the field outside the wires cancels. If the fields are of unequal magnitude, note that the overall field will be dominated by the field of the wire carrying the larger current.
Figure 5.19. The magnetic field of two wires carrying equal currents in opposite directions.

Figure 5.20. The magnetic field of two wires carrying unequal currents in opposite directions.
Think About It

Using the Lorentz force law, determine if the force between two parallel current carrying wires is attractive or repulsive. Does it make any difference if the currents are traveling in the same direction or in opposite directions?

Hint: When you consider the force on one wire, you should: 1) find the magnetic field of the second wire on the first wire (ignoring the field of the first wire as its field doesn’t contribute to the force it feels) and 2) use \( \vec{F} = q\vec{v} \times \vec{B} \) to find the direction of the force on a charge within the wire.

Does it make any difference if you think of the charges as positive charges moving in the direction of the current or negative charges moving opposite the direction of the current?

Things to remember:
- As with charges, it is best to consider first the near field and the far field.
- If two parallel wires carry current in the same direction, the intermediate field typically has peanut-shaped field lines.
- The intermediate field lines will be typically closer to a wire on one side than the other. Look at whether the fields of the individual wires are in the same direction or in opposite directions to determine which side has the larger field.

5.10 Field Contours

The field contours for each type of field are most easily drawn by first sketching the field lines, then drawing the surface perpendicular to those lines. Figs. 5.21 – 5.24 show typical examples of these.

Note that, with the flat representation for the electric field, there are several similarities between the shapes of the field lines of electric field and the field contours for magnetic fields. Conversely, there are similarities between the field contours of electric fields and the field lines of magnetic fields. Remember, though, that the similarities are only superficial if we view the systems in three dimensions.
Figure 5.21. Electric field lines and contours of two positive charges.

Figure 5.22. Magnetic field lines and contours for two wires carrying current out of the screen.
Figure 5.23. Electric field lines and contours for two opposite charges.

Figure 5.24. Magnetic field lines and contours for two currents in opposite directions.
Things to remember:
• In flat diagrams the electric field lines resemble the magnetic field contours and the magnetic field lines resemble the electric field contours.

5.11 A Summary of the Rules

1. Since electric field lines and magnetic field contours carry the geometry of the field laws, we concentrate on drawing these. Remember that, although the drawings look similar for both cases, they represent much different things in three dimensions.

2. The field lines and contours in the near field and the far field are those of point charges or current-carrying wires.

3. In between these extremes remember:
   - Field lines (contours) never cross.
   - Field lines (contours) are closer together where the field is stronger.
   - Field lines (contours) originate and end on charges (wires).

4. Draw electric field contours and magnetic field lines perpendicular to the electric field lines and magnetic field contours. Try to space them so that they are closer together where the field is stronger.

5. Practice drawing field lines and contours for a number of simple arrangements of charges and currents.