Chapter 12 – Inductors and AC Circuits

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History

Concepts from previous physics and math courses that you will need for this chapter:
- electric field see Reminders
- magnetic field see Reminders
- threads see Reminders
- stubs see Reminders

When you finish this chapter, you will be able to:

- describe
- explain
- explain

12.0 Introduction

Intro
12.1 An Inductor in a DC Circuit

A coil of wire placed in a circuit is called an inductor. A number of inductors are illustrated in Fig. 12.1. While inductors can take on a variety of forms, the simplest inductor is just a solenoid.

Figure 12.1. A variety of inductors.

Let’s consider an inductor in the simple series DC circuit illustrated in Fig. 12.2.

![Figure 12.2. A DC circuit with an inductor.]

We know that the current passing through the inductor will create a magnetic field and that the inductor will have a small resistance, but other than that, the inductor is simply a length of wire. If we ignore its resistance, there will be a current given by Ohm’s Law: \( i = \frac{V}{R} \).

Now let’s take a magnet and bring near the inductor, as shown in Fig. 12.3.

![Figure 12.3. A DC circuit with an inductor and a magnet.]

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There will be field lines from the magnet coming down into the inductor. Since the field is increasing, the inductor will create an induced magnetic field that will point upward. This will cause a current to flow in the circuit. (Depending on your point of view, you may see the coils wound one way or the other. But, if we assume we’re looking somewhat downward on the coils, current will flow from the top of the inductor toward the positive terminal of the battery.) If the south pole of the magnet is brought near the inductor in the same fashion, current will flow in the opposite direction. If the magnet is stationary, it doesn’t affect the circuit at all. In other words, an inductor will change a circuit if there’s a changing magnetic field passing through it, so that an induced current will be produced.

Things to remember:
• In a DC circuit, an inductor normally behaves just as a long segment of wire.
• If there is a changing magnetic field in an inductor, current will be induced in the circuit.

12.2. An Inductor in an AC Circuit

Now let’s attach the inductor and resistor to an AC power supply, as shown in Fig. 12.4. Because the power supply is changing sinusoidally, the magnetic field produced by the inductor also varies sinusoidally. And since the magnetic field in the inductor varies in time, an induced EMF is produced in the coil. This process is called “self induction.”

![Figure 12.4. An AC circuit with an inductor.](image)

The easiest way of describing self induction is to find the EMF produced by the inductor. We will ignore any resistance in the inductor’s coils, so the voltage across the inductor is just the induced EMF. Knowing the EMF will allow us to calculate how the current in the circuit behaves in time.

Now let’s use Faraday’s Law to see just what in inductor does in a circuit. Let’s assume we have a solenoidal inductor with a cross sectional area \( A \) and \( N \) turns of wire over a length \( \ell \). We define \( n = N/\ell \) to be the number of turns per unit length as we did in Chapter 8. Putting this all together, we get:
\[ B = \mu_0 ni \]
\[ \Phi_b = BA = \mu_0 niA \]
\[ N\Phi_b = N\mu_0 niA = \mu_0 n^2 i\ell A \]
\[ \varepsilon = -N \frac{d\Phi_b}{dt} = -\mu_0 n^2 \ell A \frac{di}{dt} \]

As we can see, the EMF equals some quantities that depend only on the size and shape of the solenoid and on the rate at which current changes in circuit. This isn’t too surprising, as the rate flux changes in the solenoid must be tied to the rate current changes in the circuit.

For convenience, we lump all of the geometrical factors into one term called the “inductance” of the inductor. The inductance of a solenoidal inductor is, then

(12.1 Inductance of a solenoid) \[ L = \mu_0 n^2 \ell A . \]

In general, inductance can be defined from the relationship

\[ \text{Inductance} \]
\[ (12.2) \]
\[ V_L = -L \frac{di}{dt} \]

where:
- \( V_L \) is the voltage across the inductor measured in volts (V).
- \( L \) is the inductance of the inductor measured in henries (H).
- \( I \) is the current through the inductor measured in amperes (A).

As with many equations, the sign of the inductance equation can be confusing. The rule is that voltage is positive if it tends to drive current in the direction current is already flowing. This is just a consequence of Lenz’s Law. If current in the circuit is decreasing, the voltage of the inductor pushes (+) charge in the direction of the current to increase the current to oppose its decrease. If the current is increasing, the inductor pushes charge against the current in order to reduce the current and oppose its increase.

Note the similarity between the equations for voltage across a resistor, a capacitor, and an inductor:

\[ V = Ri = R \frac{dq}{dt} \]
\[ V = \frac{1}{C} q \]
\[ V = -L \frac{di}{dt} = -L \frac{d q}{dt^2} \]
It may not take a great deal of imagination to believe that inductance adds like resistance rather than capacitance in series and parallel combinations. To be more rigorous, we can see that the voltages of two inductors in parallel must be the same and that \( \frac{di}{dt} \) as well as \( i \) must the same for two inductors in series.

In series:

\[
V = V_1 + V_2 \\
-L \frac{di}{dt} = -L_1 \frac{di}{dt} - L_2 \frac{di}{dt} \\
\Rightarrow L = L_1 + L_2
\]

In parallel:

\[
i = i_1 + i_2 \\
\Rightarrow \frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} \\
\frac{V}{L} = \frac{V}{L_1} - \frac{V}{L_2} \\
\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}
\]

Things to remember:

- Inductance is defined by the equation: \( V_L = -L \frac{di}{dt} \)
- Inductance depends on the geometry of the inductor, not on the current, etc.
- Inductance adds like resistance in series and parallel combinations.

12.3 Energy in Inductors and Magnetic Fields

Let’s take a very simple circuit consisting only of an inductor and an AC power supply, as illustrated in Fig. 12.5.
Figure 12.5. An AC circuit with an inductor.

Not worrying too much about signs, we know that the voltage across the power supply must equal the voltage across the inductor:

\[ \varepsilon = L \frac{di}{dt}. \]

and the power provided by the power supply must be

\[ P = i\varepsilon = iL \frac{di}{dt} = \frac{di}{dt} \frac{1}{2} Li^2. \]

Since power is the rate of change of energy, and the only energy is the potential energy of the inductor, we must conclude:

(12.3 Energy stored in an inductor) \[ U_L = \frac{1}{2} Li^2. \]

If we take the special case of a solenoidal inductor, we can write the energy as:

\[ U_L = \frac{1}{2} Li^2 \]
\[ = \frac{1}{2} \mu_0 n^2 A l i^2 \]
\[ \frac{U_L}{A l} = \frac{1}{2} \mu_0 n^2 i^2 \]
Since the magnetic field is \( B = \mu_0 ni \) and the volume of the solenoid is \( vol = A\ell \), we can write the energy density in the inductor as:

\[
(12.4 \text{ Energy density in a magnetic field}) \quad u = \frac{U}{vol} = \frac{1}{2\mu_0} B^2.
\]

Note that these equation bear strong resemblance to the equations for energy stored in a capacitor and the energy density of the electric field:

\[
U_c = \frac{1}{2} CV^2, \quad u = \frac{U}{vol} = \frac{1}{2} \varepsilon_0 E^2.
\]

Things to remember:

• The energy stored in an inductor is \( U_L = \frac{1}{2} Li^2 \).

• The energy density of a magnetic field is \( u = \frac{U}{vol} = \frac{1}{2\mu_0} B^2 \).

12.4. LR Circuits

Let’s return again to a simple circuit containing a battery, a resistor, and an inductor all connected in series; however, now let’s add a switch to the circuit.

![Series LR Circuit Diagram](image)

Figure 12.6. A series LR circuit.

Initially there is no current and no magnetic field in the inductor. As soon as the switch is closed, current starts flowing from the battery and magnetic starts being produced in the inductor. The inductor tries to oppose change in the system. That is, it produces an induced current that opposes the current from the battery and opposes the creation of a magnetic field in the inductor.

Let’s see if we can find the current as a function of time in the LR circuit. To do this, we apply Kirchoff’s Loop Law to the circuit. One thing we need to be careful of is to get the signs
correct in the loop equation. Just after the switch is closed, we know the EMF of the inductor opposes that of the battery, so we can put + and – signs on the circuit elements as in Fig. 12.7.

![LR Circuit Diagram](image)

**Figure 12.7.** An LR circuit with + and – signs of the voltages added.

Now, we can write out the loop equation:

\[
V - L \left| \frac{di}{dt} \right| - iR = 0
\]

We need to remove the absolute value removed. To do this, we have to ask whether the current is increasing in time or decreasing in time. It may not be obvious, but it turns out that the initial current is zero and it rises to a final value of \( i = \frac{V}{R} \), the current that would flow if there were no inductor. Hence:

\[
\frac{di}{dt} > 0
\]

\[
\Rightarrow V - L \frac{di}{dt} - iR = 0
\]

We might guess (and since I know the answer already, the guess is correct) that the solution to this equation would be similar to the equation of a charging capacitor. So let’s try:

\[
i(t) = \frac{V}{R} \left( 1 - e^{-t/\tau} \right)
\]

\[
\Rightarrow V - L \frac{V}{R \tau} e^{-t/\tau} - V \left( 1 - e^{-t/\tau} \right) = 0
\]

\[
V - L \frac{V}{R \tau} e^{-t/\tau} - V + Ve^{-t/\tau} = 0
\]

\[
\Rightarrow -L \frac{V}{R \tau} e^{-t/\tau} + Ve^{-t/\tau} = 0
\]

\[
\Rightarrow \tau = \frac{L}{R}
\]
We see then that the current obeys an exponential equation, much as is the charge in a charging capacitor; however the time constant is now $\tau = L/R$. If the time constant is large, it takes a long time for the current to reach its maximum value. It makes sense that a large inductor would be better able to oppose the battery’s current and that it would take a relatively large time for the current to increase to its final value. The dependence of the time constant on resistance may be a little harder to understand intuitively. However, we think of the inductor as creating an induced current that continues to flow opposite the battery. The smaller the resistance, the longer time it takes for the induced current to die out leaving only the current of the battery.

There is one more type of LR circuit we can consider, that shown in Fig. 12.8 below.

![Figure 12.8. A variation of the LR circuit.](image)

In this circuit, as switch is initially in position 1 and a steady-state current $i = V/R$ is flowing through the inductor. Then at time $t=0$, the switch is moved to position 2, removing the battery from the circuit. The inductor tries to keep current flowing through the circuit as long as it can. The inductor then acts like a battery pushing current around the circuit in the original direction, as shown in Fig. 12.9.

![Figure 12.9. The circuit of Fig. 12.8 with the switch flipped to position 2.](image)

The loop equation for the circuit with the switch in position 2 is

$$L \frac{di}{dt} - iR = 0.$$
The current is now decreasing, so to remove the absolute values we need to do the following:

\[
\frac{di}{dt} = -0
\]

\[-L \frac{di}{dt} - iR = 0\]

This time we might expect the current to be an exponentially decreasing function of time.

\[
i(t) = \frac{V}{R} e^{-t/\tau}
\]

\[+ L \frac{V}{R\tau} e^{-t/\tau} - R \frac{V}{R} e^{-t/\tau} = 0\]

\[\Rightarrow \tau = \frac{L}{R}\]

Just as with RC circuits, the same time constant governs both \( e^{-t/\tau} \) and \( 1 - e^{-t/\tau} \) equations.

Things to remember:

- In LR circuits, current changes in time either by \( e^{-t/\tau} \) or by \( 1 - e^{-t/\tau} \).
- The LR time constant is \( \tau = \frac{L}{R} \).

### 12.5. LC Circuits and Phases

The next thing we can do is consider what happens when we charge a capacitor to a voltage \( V_0 \) and connect it in series with an inductor, as shown in Fig. 12.9.

![Figure 12.9. An LC circuit.](image)

At first glance, it might not seem that this circuit would be much different than a circuit with a battery; however, the current changes as the capacitor discharges, so an induced is produced on the inductor. We can qualitatively guess what should happen with this circuit either by consider the current and charge or by considering energy. Since both are instructive in different ways, we'll look at each in turn.
First, let’s think about charge and current. We begin by connecting a charged capacitor to the inductor. As time progresses, the circuit goes through the following stages:

1) The capacitor begins to discharge, but the inductor opposes the flow of current.
2) The current from the capacitor increases as the capacitor discharges.
3) The capacitor is fully discharged but current is flowing through the circuit. The inductor keeps current in the same direction.
4) The capacitor begins to charge in the opposite direction.
5) The capacitor has charge $Q$ in the opposite direction and current ceases to flow.

![Figure 12.10. Successive stages of the capacitor discharging and charging again in the opposite direction.](image)

Note that since there is no resistance in this circuit, the charge oscillates back and forth indefinitely.

Now let’s think about the same process in terms of energy. We’ll follow the same stages as before:

1) All the energy is in the electric field of the capacitor.
2) As current begins to flow, some of the capacitor’s energy is transferred to the magnetic field of the inductor.
3) The capacitor is fully discharged and all the energy in the system is in the inductor. This implies that the current reaches a maximum at this point.
4) The capacitor begins recharging and some of the energy is transferred back to the capacitor.
5) All the energy goes back to the electric field of the inductor.

At this point we could guess that the solution to the problem must be something like $q(t) = CV_0 \sin \omega t$, but we don’t know what $\omega$ is. Let’s see if we can apply Kirchoff’s Loop Law as we did before. The tricky part is to get the signs right. To do that, all we have to do is find any time where the signs are all consistent and write down the equation at that time. The signs at other times will be consistent with that time. At this point, we need to establish a sign convention for voltages. The reason we need to do this is that signs can quickly become confusing when the direction of the current is constantly changing. Our basis for the sign convention is that we want to use Ohm’s Law $V=IR$ the same way in AC circuits as in DC.
circuits. Note that the voltage across a resistor is taken to be positive when the voltage *opposes* current flow. We then use this same convention for capacitors and inductors:

**Sign convention for voltages in AC circuits.**

Define a positive sense for current. Voltage across a resistor, capacitor, or inductor is positive if it pushes current in the negative direction and negative if it pushes current in the positive direction.

There are some very important consequences to this sign convention, so we should take a little while to go over these. Let’s consider the case of resistors, capacitors, and inductors individually.

For resistors, the sign convention is quite simple. When the current is positive, the resistor pushes current in the negative direction, so the voltage is positive. When the current is negative, it pushes current in the positive direction, so the voltage is negative. We can write

\[ V = iR. \]

![Figure 12.11. Current through and the voltage across a resistor in an AC circuit.](image)

We say that the voltage across a resistor is “in phase” with the current through the resistor. That is, the peaks and valleys of the two functions occur at the same time.

For inductors, we know the induced voltage will oppose the change in current. When the current is positive and increasing, the induced EMF will oppose the increase, pushing charge in the negative direction. Let’s consider the signs for the voltage in each of the possible combinations of positive and negative current and increasing and decreasing current. The one tricky part of the table is to remember that if the current is negative and \( di/dt \) is positive, the current is getting more positive – meaning that the magnitude of the current is dropping. Similarly if \( i \) is negative and \( di/dt \) is also negative, the current is getting smaller (more negative) so its magnitude is getting larger. Be sure you think about that a bit before you go on.
<table>
<thead>
<tr>
<th>Current</th>
<th>$\frac{di}{dt}$</th>
<th>Current Magnitude</th>
<th>Induced Current Direction</th>
<th>Inductor Voltage</th>
</tr>
</thead>
<tbody>
<tr>
<td>positive</td>
<td>positive</td>
<td>increasing</td>
<td>negative</td>
<td>positive</td>
</tr>
<tr>
<td>positive</td>
<td>negative</td>
<td>decreasing</td>
<td>positive</td>
<td>negative</td>
</tr>
<tr>
<td>negative</td>
<td>positive</td>
<td>decreasing</td>
<td>negative</td>
<td>positive</td>
</tr>
<tr>
<td>negative</td>
<td>negative</td>
<td>increasing</td>
<td>positive</td>
<td>negative</td>
</tr>
</tbody>
</table>

The important thing to note about this table is that the inductor voltage is positive when the slope of the current is positive and the inductor voltage is negative when the slope of the current is negative. Because of this, we can write:

\[
V = +L \frac{di}{dt}.
\]

This equation is quite confusing because the sign seems to be reversed from our earlier result, but it the consequence of our convention that positive voltage causes current to flow in the negative direction.

![Figure 12.12. Current through and the voltage across an inductor in an AC circuit.](image)

Think About It

Look at Fig. 12.12 and convince yourself that the voltage across the inductor is positive whenever the slope of the current is positive.
In this case we say that the voltage across an inductor “leads the current by $90^\circ$,” or that the “phase angle is $+90^\circ$.” Note that the phase angle is the angular difference between the maximum voltage and the maximum current, as shown by the arrow in Fig. 12.12.

Capacitors are just a little harder. With DC circuits, we always thought of the charge on a capacitor as positive. With AC circuits we can no longer do that. By our sign convention, we must take the charge on a capacitor to be positive when it tends to drive charge against the current. But what we really want to know is what that means in terms of current. If the charge on a capacitor is positive, the capacitor voltage is positive. If current is decreasing, then current must be flowing in the negative direction. Let’s make a table of all such results:

<table>
<thead>
<tr>
<th>Capacitor Charge</th>
<th>Capacitor Voltage</th>
<th>Current Direction</th>
<th>$\frac{dq}{dt}$ or $\frac{dV}{dt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>positive</td>
<td>positive</td>
<td>negative</td>
<td>negative</td>
</tr>
<tr>
<td>positive</td>
<td>positive</td>
<td>positive</td>
<td>positive</td>
</tr>
<tr>
<td>negative</td>
<td>negative</td>
<td>negative</td>
<td>negative</td>
</tr>
<tr>
<td>negative</td>
<td>negative</td>
<td>positive</td>
<td>positive</td>
</tr>
</tbody>
</table>

This table tells us that $i = +\frac{dq}{dt}$. But since the charge and voltage have the same sign, we see that whenever the voltage has a negative slope, the current must be negative. Similarly, whenever the voltage has a positive slope, the current must be positive. These results lead to the graph shown in Fig. 12.13.

![Figure 12.13. Current through and the voltage across a capacitor in an AC circuit.](image)

In this case we say that the voltage across a capacitor “lags the current by $90^\circ$,” or that the “phase angle is $-90^\circ$.” Again, the phase angle is the angular difference between the maximum voltage and the maximum current.
Now let’s return to our simple LC circuit. For simplicity, we take a time shortly after current begins to flow from the capacitor, as shown in Fig. 12.11. Let’s take positive current to be in the clockwise direction. The charge on the capacitor tends to drive current in the positive direction, so the capacitor charge and voltage will both be negative initially (by our confusing sign convention). Since the current is getting larger in magnitude, the EMF on the inductor will drive the current in the negative direction, and hence be positive.

Figure 12.14. An LC circuit shortly after the capacitor begins discharging.

In this case, we can write the loop equation as:

\[ -|V_c| + |V_L| = 0 \]

\[ -\frac{q}{C} + L\frac{di}{dt} = 0 \]

We need to get rid of the absolute value notation. In Fig. 12.11 the charge on the capacitor is negative and increasing (getting less negative) and the current is positive and increasing. Thus, we have:

\[ q < 0 \]
\[ \frac{dq}{dt} > 0 \]
\[ i > 0 \]
\[ \frac{di}{dt} > 0 \]

To keep all the signs consistent, we must have:

\[ i = +\frac{dq}{dt} \]
\[ \frac{di}{dt} = +\frac{d^2q}{dt^2} > 0 \]

\[ \Rightarrow +\frac{q}{C} + L\frac{d^2q}{dt^2} = 0 \]
This then leads to the differential equation for LC circuits:

\[\frac{d^2 q}{dt^2} = -\frac{1}{LC} q\]  

(12.6)

As you may easily verify, the solution to this equation must be a combination of sines and cosines. Since we want charge to be a minimum at time \(t=0\), we choose \(q(t) = -CV_0 \cos \omega t\). Then:

\[-\omega^2 CV_0 \cos \omega t = -\frac{1}{LC} CV_0 \cos \omega t\]

\[\Rightarrow \omega = \frac{1}{\sqrt{LC}}\]

This tells us the frequency at which the charge on the capacitor, the current in the circuit, the energy in the circuit, the voltage on the capacitor, the voltage on the inductor – everything in the circuit – oscillates.

Once we know the charge on the capacitor, we can find anything we want.

\[\omega = \frac{1}{\sqrt{LC}} \quad f = \frac{1}{2\pi \sqrt{LC}}\]

\[q(t) = -CV_0 \cos \omega t\]

\[V_c = \frac{q(t)}{C} = -V_0 \cos \omega t\]

\[i = \frac{dq}{dt} = CV_0 \omega \sin \omega t = V_0 \sqrt{\frac{C}{L}} \sin \omega t\]

\[V_L = +L \frac{di}{dt} = LCV_0 \omega^2 \cos \omega t = V_0 \cos \omega t\]

In Fig.12.15, we plot the current and the voltages across the capacitor and the inductor as a function of time.
Figure 12.15. Current and voltages in an LC Circuit.

Note how the capacitor’s voltage peaks after the current, but the inductor’s voltage peaks before the current, just as we had suggested above. A convenient way to remember these phase relationship is to use the mnemonic device below.

**ELI the ICE man**

In an inductor, the EMF ($V_L$) leads the current by 90°.

In a capacitor, the current leads the EMF ($V_C$) by 90°.

Things to remember:

- An LC circuit oscillates at an angular frequency $\omega = \frac{1}{\sqrt{LC}}$.
- Energy is transferred back and forth between the electric field of the capacitor and the magnetic field of the inductor as the circuit oscillates.
- ELI the ICE man – and its meaning.
- Know how to derive Kirchoff’s loop equation, Eq 12.6. You may be a bit cavalier about signs.
12.6. Phasors

The word “phasor” is short for “phase vector.” It is a way to represent a sine or cosine function graphically. If you have taken Physics 123, you may have used phasors to analyze the interference of light through slits. In this course, phasors are very helpful in visualizing and analyzing AC circuits.

In AC circuits, currents and voltages are all sinusoidal functions. The general mathematical form of such a function is:

\[ A(t) = A_0 \sin(\omega t + \phi) \]

where

- \( A(t) \) is the value of \( A \) (generally a current or voltage) at time \( t \).
- \( A_0 \) is the maximum value of \( A \).
- \( \omega \) is the angular frequency in rad/s.
- \( \phi \) is the phase angle.

A phasor is a vector which has length \( A_0 \) and is directed at an angle \( \theta = \omega t + \phi \) to the \( x \) axis, as shown in Fig. 12.16.

![Figure 12.17. A phasor representing the function \( A(t) = A_0 \sin(\omega t + \phi) \).](http://www.physics.byu.edu/faculty/rees/220/Graphics/phasorB.gif)

As any other vector the phasor \( \vec{A} \) can be expressed in terms of components:

\[ \vec{A} = A_0 \cos(\omega t + \phi) \hat{x} + A_0 \cos(\omega t + \phi) \hat{y}. \]

From this we can see the relation between the phasor and the function is that the function is just the \( y \) component of the phasor.

The angle of such a phasor changes in time, so as time progresses, the phasor rotates about the origin at an angular velocity \( \omega \). This is illustrated in Fig. 12.18 or in an animated version on the course website at [http://www.physics.byu.edu/faculty/rees/220/Graphics/phasorB.gif](http://www.physics.byu.edu/faculty/rees/220/Graphics/phasorB.gif).
Let’s assume that we have an AC circuit with a current given by the equation $i(t) = i_0 \sin(\omega t)$. Assume that both $i_0$ and $\omega$ are known. We wish to then construct phasors for the voltages across resistors, inductors, and capacitors.

A. Resistors

In order to construct a phasor for the voltage across any circuit element, we need to know the magnitude and the angle of the phasor. This is easy for a resistor, as we only need Ohm’s Law and the knowledge that the phase angle for a resistor is 0°. We then have:

$$V_R = i(r)R = i_0R\sin(\omega t)$$
$$V_{R0} = i_0R$$

where

- $V_{R0}$ is the maximum voltage across the resistor in volts (V).
- $i_0$ is maximum current through the resistor in amperes (A).
- $R$ is the resistance in ohms (Ω).

We can draw the phasor when the current is at any angle. For simplicity, let’s draw it for time $t=0$.

An animated version of this can be found at
B. Inductors

Now we can go through the same process for inductors. We know the angle of the voltage phasor as it is $90^\circ$ ahead of the current. The magnitude of the phasor comes from the relationship:

$$V_L = L \frac{di}{dt} = \omega L i_0 \cos(\omega t)$$

$$V_{L,0} = i_0 \omega L$$

Note that this equation looks a lot like Ohm’s Law. Even though an inductor has no resistance and no energy loss, the inductance offers an effective resistance to limit the flow of current through a circuit. We call this effective resistance the “inductive reactance” and write it as:

(12.5 Inductive Reactance)

$$X_L = \omega L$$

$$V_{L,0} = i_0 X_L$$

It is reasonable that the effective resistance for an inductor is $\omega L$ since higher frequencies and larger inductance both lead to larger induced currents.

Then we can draw the phasors for an inductor as follows:

![Figure 12.20. The voltage and current phasors for an inductor.](http://www.physics.byu.edu/faculty/rees/220/Graphics/LPhasor.gif)
C. Capacitors

Finally, we come to capacitors. The angle of the voltage phasor as it is 90° behind the current. Since $i = dq/dt$, the magnitude of the phasor comes from the relationship:

$$V_c = \frac{q}{C} = L \int i dt = -\frac{1}{\omega C} i_0 \cos(\omega t)$$

$$V_{L0} = i_0 \frac{1}{\omega C}$$

As with an inductor, a capacitor has no resistance and no energy loss, but it does produce an effective resistance in a circuit. We call this effective resistance the “capacitive reactance” and write it as:

$$X_C = \frac{1}{\omega C}$$

$$V_{C0} = i_0 X_C$$

(12.6 Capacitive Reactance)

To understand this relationship, we should remember that a capacitor offers resistance in a circuit when it charges and opposes current flows. The larger the charge the capacitor develops, the larger its effective resistance in a circuit. If frequency is very high, a capacitor has little chance to charge before the current reverses direction, so it offers little resistance to current. Similarly, if the capacitance is large, a large amount of charge can collect on a capacitor plates without increasing the voltage much.

Then we can draw the phasors for an inductor as follows:

![Figure 12.20. The voltage and current phasors for an inductor.](http://www.physics.byu.edu/faculty/rees/220/Graphics/LPhasor.gif)
Things to remember:
• A sine wave can be represented by the projection of a phasor onto the $y$ axis.
• The length of a phasor is the amplitude of the sine wave. The angle of a phasor with respect to the $x$ axis is the angular argument (the phase angle) of the sine function.
• Phasors of waves can be added as vectors to produce the sum of two sine functions.
• For AC circuits, the phase angle is $\omega t$, so phasors rotate counterclockwise at an angular speed of $\omega$.
• We usually wish to construct current and voltage phasors for each circuit element.
• For resistors, the current and voltage phasors are in phase.
• For inductors, the voltage phasor is at an angle of +90° from the current phasor.
• For capacitors, the voltage phasor is at an angle of –90° from the current phasor.

12.7. Rules for AC Circuits

We can use similar rules for AC circuits as we had for DC circuits, but with small modifications to take into account the sinusoidal variation in voltages and currents.

<table>
<thead>
<tr>
<th>Rules for AC Circuits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. If two circuit elements are in series, they have the same current phasor and their voltage phasors add as vectors.</td>
</tr>
<tr>
<td>2. If two circuit elements are in parallel, they have the same voltage phasor and their current phasors add as vectors.</td>
</tr>
<tr>
<td>3. The sum of current phasors into a junction equals the sum of current phasors out of a junction.</td>
</tr>
<tr>
<td>4. The sum of voltage phasors from circuit elements around a loop is the sum of voltage phasors from EMFs around a loop. (Note that they don’t sum to zero because of our standard definition of positive voltage. That is, voltage phasors for circuit elements are voltage drops, whereas voltages for EMFs are voltage gains.)</td>
</tr>
</tbody>
</table>

In order to see how to apply these rules, let’s take a specific example.

Example 12.1. An AC circuit with series and parallel elements.
First, we want to give numerical values for a number of the quantities in the problem:

\[
\begin{align*}
\omega & = 1.250 \times 10^5 \, \text{Hz} \\
L & = 3.200 \times 10^{-5} \, \text{H} \\
C & = 2.000 \times 10^{-6} \, \text{F} \\
R_1 & = 2.000\Omega \\
R_2 & = 1.000\Omega \\
i_2 & = 3.000A
\end{align*}
\]

We wish to find the voltage of the power supply – and all the other voltages and currents in the circuits while we’re at it.

When we worked with DC circuits using Kirchoff’s Laws, the first thing we did was assign a direction for the current. With AC circuits, we need to define the direction we take to be positive. With a single EMF, we should think of the power supply as a battery and draw the currents so they are consistent with flow of current from a battery. This is done in Fig. 12.21.

First, let’s draw current and voltage phasors for the inductor. For simplicity, we can draw the current phasor along the +x axis. We know the voltage phasor will be along the +y axis and that its magnitude will be:

\[
\begin{align*}
X_L & = \omega L = 4.000\Omega \\
V_{L0} & = 12.00V
\end{align*}
\]
This gives us:

Next we add a phasor for the resistor, $R_2$. Since this resistor is in series with the inductor, the share the same current phasor. The length of the voltage phasor for the resistor is:

$$V_{R2} = i_2 R_2 = 3.000V.$$  

The voltage of the resistor is in phase with the current.

Next, we add the phasors for the inductor and resistor voltage to give us the voltage phasor for the combination of the two circuit elements.
We would like to find the magnitude of $\vec{V}_{LR}$, the angle that $\vec{V}_{LR}$ makes with respect to the $x$ axis, and the components of $\vec{V}_{LR}$. This can be accomplished by simple geometry:

$$V_{LR0} = \sqrt{V_{L0}^2 + V_{R0}^2} = 12.37V$$

$$\tan \phi = \frac{V_{L0}}{V_{R0}} = \frac{4}{1}, \quad \phi = 75.96^\circ$$

$$\vec{V}_{LR} = V_{LR0} \cos \phi \hat{x} + V_{LR0} \sin \phi \hat{y} = V_{R0} \hat{x} + V_{L0} \hat{y} = 3.000V \hat{x} + 12.00V \hat{y}$$

Before we leave the resistor and inductor, there is one more thing we can find, the impedance of the $LR$ combination.

$$Z_{LR} = \frac{V_{LR0}}{i_2} = 4.123\Omega$$

or

$$Z_{LR} = \frac{\sqrt{V_{L0}^2 + V_{R0}^2}}{i_2} = \frac{\sqrt{i_2^2 X_L^2 + i_2^2 R^2}}{i_2}$$

$$Z_{LR} = \sqrt{X_L^2 + R^2} = 4.123\Omega$$

Now that we’ve found the voltage, current, and impedance for the branch of the circuit with $i_2$, we turn our attention to the capacitor. Since the capacitor is in parallel with the inductor-resistor combination, we know the capacitor’s voltage phasor will be the same as $\vec{V}_{LR}$. In the capacitor (ICE), the current leads the voltage, so we know the direction of the current phasor, $\vec{i}_3$. We can find the magnitude of the current phasor by considering the voltage phasors:
\[ X_c = \frac{1}{\omega C} = 4.000 \Omega \]
\[ V_{C0} = i_3 X_c = V_{LR0} = 12.37 \text{V} \]
\[ i_3 = 3.092 \text{A} \]

Now, let’s draw the phasor diagram for the capacitor.

For the next step, we note that the currents in the two parallel branches, \( \vec{i}_2 \) and \( \vec{i}_3 \), add as phasors to give the total current, \( \vec{i}_1 \).
Let’s algebraically work out the components of $\vec{i}_2$ and $\vec{i}_3$. We know that the angle between $\vec{i}_3$ and the negative x axis is $90^\circ - \phi = 14.04^\circ$, and the length of the phasor is 3.092A, so:

$$\vec{i}_3 = -(3.092A) \cos(14.04^\circ) \hat{x} + (3.092A) \sin(14.04^\circ) \hat{y}$$
$$= -3.000A \hat{x} + 0.750A \hat{y}$$

$$\vec{i}_i = \vec{i}_2 + \vec{i}_3 = 0.750A \hat{y}$$

Note that we add phasors exactly the same way we add any other vectors.

Now that we have the current through $R_i$, we can easily find the voltage phasor for this resistor:

$$\vec{V}_R = \vec{i}_i R_i = 1.500V \hat{y}.$$ 

We can solve for the EMF since the voltage phasors around any closed loop is zero. Keeping in mind the sign convention for voltages across circuit elements is opposite that for the power supply, we have:

$$\vec{\epsilon} - \vec{V}_R - \vec{V}_c = 0$$
$$\vec{\epsilon} = \vec{V}_R - \vec{V}_c$$
$$= 1.500V \hat{y} + 3.000V \hat{x} + 12.00V \hat{y}$$
$$= 3.000V \hat{x} + 13.50V \hat{y}$$

$$\epsilon_0 = \sqrt{(3.000V)^2 + (13.50V)^2} = 13.83V$$

Finally, we would like to know the phase angle between the current $\vec{i}_i$ and the EMF, $\vec{\epsilon}$. In this case, it is easy to find the angle with simple geometry, since the current is in the y direction.
However, I’ll use a little trick that is handy when we want to find the angle between arbitrary vectors. For two general vectors:

\[ \vec{A} \cdot \vec{B} = AB \cos \phi \]

\[ \cos \phi = \frac{\vec{A} \cdot \vec{B}}{AB} \]

Applying this rule to our vectors:

\[ \cos \phi = \frac{\vec{i}_i \cdot \vec{e}}{\vec{i}_i \vec{e}} = \frac{0.750 \times 13.50}{0.750 \times 13.83} \]

\[ \phi = \pm 12.53^\circ \]

Since \( \vec{i}_i \) leads \( \vec{e} \), the phase angle is negative (more like a capacitor) and we conclude that

\[ \phi = -12.53^\circ \]

Things to remember:
- Know the rules outlined in the box at the beginning of this section.

### 12.8 Power in AC Circuits

We know that in a DC circuit \( P = iV \). In an AC circuit, this same result must hold; however, power will be a function of time. First, let’s consider a resistor. Assuming a sinusoidal current, we know the voltage across the resistor will be in phase with the current. Hence:

\[ P(t) = i(t)V(t) = i_0V_0 \sin^2 (\omega t) . \]

Although this is true, we often find that it is more useful to know the average power dissipated by the resistor over one full cycle (or many full cycles). To find the average power, we can first think of taking \( N \) samples over one full cycle.

\[ \bar{P} = \frac{1}{N} \sum_{j=1}^{N} P_j = \frac{i_0V_0}{N} \sum_{j=1}^{N} \sin^2 (\omega t_j) \]

This, of course, is just an approximation to the average, since \( N \) would have to be infinite for the average to be exact. We actually can take the sum over an infinite number of terms by turning the sum into an integral. To do this, let’s divide one cycle into \( N \) time intervals each of length \( \Delta t \), so that \( \Delta t = T / N \) where \( T \) is one full period. Then

\[ \bar{P} = \frac{i_0V_0}{N\Delta t} \sum_{j=1}^{N} \sin^2 (\omega t_j) \Delta t \rightarrow \frac{i_0V_0}{T} \int_0^T \sin^2 (\omega t) dt \]
The integral is a standard one for calculus courses, but we don’t actually do it here. It is sufficient to know that the result is just $T/2$. This then gives us:

\[
\bar{P} = \frac{1}{2} i_0 V_0 = \frac{1}{2} i_0^2 R = \frac{V_0^2}{2 R}.
\]

This equation brings up a practical question. When we say an AC power supply provides a given voltage, what number should we use for the voltage? We could use the average voltage, but that’s just zero. Another good idea would be to average the absolute value of the voltage over one cycle. The drawback to this is that absolute values are rather messy mathematical functions. The solution physicists chose was based on average power. They suggested that a good definition for effective voltage is the voltage that would yield the same power as a DC voltage. That is, we let:

\[
\bar{P} = \frac{V_{\text{eff}}^2}{R} = \frac{V_0^2}{2R}.
\]

\[\Rightarrow V_{\text{eff}} = \frac{1}{\sqrt{2}} V_0 \]

Mathematically, the way we obtain this effective voltage is essentially what we did to find the average power: we square the voltage, average the squared voltage over one complete cycle, and then take the square root. Because of this, the effective voltage is called the “root-mean-square voltage,” or just the “rms voltage.” So, when we say that our outlets provide 115 VAC, what we really mean is that the rms voltage is 115 V.

Think About It

What is the peak voltage in your AC power outlet?

Of course we can define rms currents in a similar fashion to rms voltages, and the power in terms of these rms quantities.

\[
\begin{align*}
V_0 &= \sqrt{2} V_{\text{rms}} \\
i_0 &= \sqrt{2} i_{\text{rms}} \\
\bar{P}_r &= i_{\text{rms}} V_{\text{rms}}
\end{align*}
\]

So far, we have only considered the power dissipated by a resistor. We also want to be able to calculate the power provided by a power supply. We can follow the same method we used above, but we do have to take into consideration that there is a phase difference between the
voltage and the current. Denoting the phase angle from the current phasor to the voltage phasor as \( \phi \) (\( \phi = +90^\circ \) for an inductor, for example), the instantaneous power is:

\[
P(t) = i(t)\epsilon(t) \\
= i_0 \sin(\omega t) \epsilon_0 \sin(\omega t + \phi) \\
= i_0 \epsilon_0 \sin(\omega t) \left[\sin(\omega t) \cos \phi + \cos(\omega t) \sin \phi\right]
\]

This leads to an average power:

\[
\overline{P} = \frac{i_0 \epsilon_0}{T} \int_0^T \sin(\omega t) \sin(\omega t) \cos \phi + \cos(\omega t) \sin \phi \, dt \\
= \frac{i_0 \epsilon_0}{T} \int_0^T \sin^2(\omega t) \, dt + \frac{i_0 \epsilon_0}{T} \int_0^T \sin(\omega t) \cos(\omega t) \, dt \\
= \frac{1}{2} i_0 \epsilon_0 \cos \phi + 0
\]

The power provided by an AC power supply, then depends on the phase angle between the current and EMF phasors. The quantity \( \cos \phi \) is called the “power factor” for the circuit. When the current is in phase in with the EMF, the power is just what it was for DC circuits, but the power provided by the power supply is less when the phase angle gets larger.

**Think About It**

What is the power factor for a resistor? a capacitor? and inductor? a capacitor? What power is dissipated in an inductor? a capacitor?

The power provided by a power supply (or really any circuit element) is given by:

\[
(12.8 \text{ AC Power, general}) \quad \overline{P} = \frac{1}{2} i_0 V_0 \cos \phi = \frac{1}{2} \mathbf{i} \cdot \mathbf{V} = i_{\text{rms}} V_{\text{rms}} \cos \phi.
\]

**Example 12.2.** Power in the circuit of Example 12.1.

What power is dissipated in each of the resistors of Fig. 12.21? What power is provided by the power supply?
Resistor $R_1$ has a resistance of $2.000 \, \Omega$ and a current of $i_{10} = 0.750A$ passes through it, so the power is $P_1 = \frac{1}{2} (0.75A)^2 (2.000\Omega) = 0.5625W$.

Resistor $R_2$ has a resistance of $1.000 \, \Omega$ and a current of $i_{20} = 3.000A$ passes through it, so the power is $P_2 = \frac{1}{2} (3.000A)^2 (1.000\Omega) = 4.5000W$.

The power provided by the power supply is (with no intermediate rounding):

$$P = \frac{1}{2} i_0 E_0 \cos \phi = 5.0625W$$

Note that the power provided by the power supply equals the total power dissipated by the resistors.

Things to remember:

- The rms values of voltage or current are the peak values divided by $\sqrt{2}$.
- The average power dissipated by a resistor in an AC circuit is $P_R = \frac{1}{2} i_{rms} V_{rms}$.
- The average power provided by an AC power source is $P = \frac{1}{2} i_0 V_0 \cos \phi$.
- $\cos \phi$ is called the power factor.

12.9. The Series LRC Circuit

The simplest and most important AC circuit we can analyze is the series LRC circuit, illustrated in Fig.12.22.
The analysis of this circuit is quite easy since all the circuit elements share the same current. We can draw a phasor diagram for the current and voltages across the inductor, capacitor, and resistor.

\[ V_{L0} = i_0 X_L \]
\[ V_{C0} = i_0 X_C \]
\[ V_{R0} = i_0 R \]
\[ \vec{e}_0 = i_0 Z \]

Since the elements are in series, we recognize that the voltages phasors of the resistor, capacitor, and inductor add together to get the total EMF, \( \vec{e} = \vec{V}_R + \vec{V}_L + \vec{V}_C \). In Fig. 12.23, we have added \( \vec{V}_L + \vec{V}_C \) first and then added that sum to \( \vec{V}_R \) to get \( \vec{e} \).

At this point, I want to introduce a little trick to simplify this particular problem. We note that every voltage in the system, including the EMF, has a common factor of \( i_0 \). If we divide this out, we can obtain a diagram for the resistance, reactances, and impedance that is quite useful. We’ll call this the “impedance diagram” of the LRC circuit. It is shown in figure 12.24.
From this diagram, we can deduce some simple equations for the total impedance and the phase angle in series LRC circuits:

\[
Z = \sqrt{(X_L - X_C)^2 + R^2}
\]

(12.9 For series LRC circuits)

\[
\tan \phi = \frac{X_L - X_C}{R}
\]

\[
\cos \phi = \frac{R}{Z}
\]

Things to remember:
- Be able to reproduce the impedance diagram for series LRC circuits, Fig. 12.24. Understand the phase relations that appear in the diagram.
- Know that \( Z = \sqrt{(X_L - X_C)^2 + R^2} \) and be able to deduce this from the impedance diagram.
- Know that \( \tan \phi = \frac{X_L - X_C}{R} \) and be able to deduce this from the impedance diagram.

### 12.10. Resonance

If we consider the impedance equation along with the equations for the inductive and capacitive reactance, we see that impedance has a rather complicated dependence on the frequency of the oscillator.
\[ Z = \sqrt{(X_L - X_C)^2 + R^2} \]
\[ X_L = \omega L \]
\[ X_C = \frac{1}{\omega C} \]

When the frequency is very small, the capacitive reactance is large and \( Z \approx X_C \). When the frequency is very large, the inductive reactance is large and \( Z \approx X_L \). \( Z \) is a minimum when \( X_L = X_C \), and \( Z \) is a minimum, the current in the circuit is a maximum. When this happens, the resistance provides the only impedance in the circuit, \( Z = R \). This condition is called resonance and is electrical analog to resonance in harmonic oscillators such as a swinging pendulum or a mass on the end of a spring.

First, let us find the frequency at which resonance is achieved:

\[ X_L = X_C \]
\[(\text{Eq. 12.10, resonance frequency})\]
\[ \omega L = \frac{1}{\omega C} \]
\[ \omega = \frac{1}{\sqrt{LC}} \]

Note that this is just the frequency at which the capacitor and inductor would oscillate if there were power supply or resistance in the circuit. Note that the condition for maximizing the current in an LRC circuit is to drive the circuit at the frequency it wants to naturally oscillate. This is similar to a swing – the maximum amplitude is obtained when we push the swing at its natural frequency of oscillation.

\[ X_L - X_C = 0 \]
\[ Z = R \]

Figure 12.25. The impedance diagram at resonance.
Things to remember:
• Resonance is when the impedance of a circuit is minimized so that the current is maximized.
• Resonance in a series LRC circuit is achieved when $X_L = X_C$.
• At resonance $Z = R$.
• The resonant frequency is the natural oscillation frequency, $\omega = \frac{1}{\sqrt{LC}}$. 