Appendix B – Relativistic Transformations

B.0 Introduction

This appendix tells how the same physical quantities are measured by observers moving at different velocities. It provides all that is necessary to understand the derivations of the thread model presented in Appendix B. If you are interested in more information on relativity including a review of matrices, please refer to Appendix D.

B.1 Definitions

First, let’s define a few important terms:

1) *a reference frame* is a three-dimensional Cartesian coordinate system with a synchronized clock available at every point in space.

2) *an inertial reference frame* (or just *an inertial* frame) is a reference frame that is not accelerating, so the standard laws of physics must apply in it.

3) *an event* is something like a little explosion – that happens at a specific location and a specific time.

4) *a rest frame* is an inertial reference frame in which an object is at rest.

5) *the fourth dimension* is time. Just as we can specify the location of an object using three Cartesian coordinates, we can add a fourth coordinate of time to specify the time of an event in addition to the spatial coordinates. In this way, we can treat time much as a spatial dimension.

6) *space-time* is the four-dimensional space that includes the three dimensions of normal (configuration) space and the one dimension of time.

7) *β (beta)* is the ratio of a velocity to the speed of light, $\beta = \frac{v}{c}$. $\beta = 0.4$ means that something – an object or a reference frame – is moving at 40% of the speed of light.

8) *γ (gamma)* is a particular function of $\beta$ that appears so regularly in equations it was given a special name. $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. When $\beta = 0$, then $\gamma = 1$. $\gamma$ remains very close to 1 until $\beta$ is greater than about 0.1. (Hence, relativistic effects are usually small for objects traveling less than about 10% of the speed of light.) As $\beta$ approaches 1, $\gamma$ becomes infinite.
To transform quantities from one reference frame to another, we first need to express them as “four-vectors.” The most fundamental of these is the space-time four-vector. As we mentioned, relativity treats time in much the same way it treats spatial variables. Therefore, we can think of an event as having four coordinates in space-time. If we call the space-time four-vector $\mathbf{r}$, it seems that we could just write it in column-vector form as:

$$
\mathbf{r} = \begin{bmatrix}
    t \\
    x \\
    y \\
    z
\end{bmatrix}
$$

The only problem with this is that the units of the vector components are not all the same. That is, time is measured in seconds, but position is measured in meters. This becomes awkward mathematically, so we get around it by multiplying the time by $c$, the speed of light. The choice of $c$ is somewhat arbitrary, but it turns out to be convenient because all observers measure the same value for $c$. Thus, we have:
r = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}.

Often we write the three spatial components together as a normal three-dimensional vector, just because we’re lazy. The meaning is identical.

1) The space-time four-vector:

\[ \mathbf{r} = \begin{bmatrix} ct \\ \vec{r} \end{bmatrix}. \]

If we know the space-time four-vector in one inertial reference frame, we can find it in another frame by using a set of equations called the Lorentz transformations. We’ll explain these transformations shortly. But first, we wish to list a few other four-vectors. All four-vectors are carefully designed to transform by the Lorentz Transformations just as the space-time four-vector. As with the space-time four-vector, we sometimes multiply components of other four-vectors by a factor of \( c \) to keep the units consistent.

2) The energy momentum four-vector:

\[ \mathbf{p} = \begin{bmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{bmatrix} = \begin{bmatrix} E \\ \vec{p} \cdot \vec{c} \end{bmatrix} \]

where \( E = m_0 \gamma_p c^2 \) and \( \vec{p} \cdot \vec{c} = m_0 \gamma_p c^2 \vec{\beta}_p \). In these equations \( m_0 \) is the rest mass, \( \vec{\beta}_p \) is the velocity (divided by \( c \)) of the particle whose momentum we are measuring, and \( \gamma_p = \frac{1}{\sqrt{1 - \beta_p^2}} \).

Note that if we multiply the equation for \( \gamma_p \) by \( E_0 = m_0 c^2 \) and square both sides, we obtain the relativistic energy-momentum relationship \( E^2 = p^2 c^2 + E_0^2 \). \( E \) is called the total energy and \( E_0 \) the rest energy. The kinetic energy is \( K = E - E_0 \).

3) The force four-vector:

\[ \mathbf{F} = \begin{bmatrix} \gamma_p \vec{\beta}_p \cdot \vec{F} \\ \gamma_p \vec{F} \end{bmatrix} = \frac{1}{E_0} \begin{bmatrix} \vec{F} \cdot \vec{p} \cdot \vec{c} \end{bmatrix} \]

where \( \vec{F} \) is the usual force in three dimensions. Note that \( \vec{F} \neq m \vec{a} \), but the more general relationship \( \vec{F} = \frac{d\vec{p}}{dt} \) is still valid. This leads to
\[ \vec{F} = E\vec{\alpha}_p + E\vec{\beta}_p \gamma_p^2 \left( \vec{\beta}_p \cdot \vec{\alpha}_p \right) \]

where \( \vec{\alpha}_p = \frac{1}{c} \frac{d\vec{p}_p}{dt} = \frac{\vec{a}_p}{c^2} \) and \( \vec{a}_p \) is the usual acceleration of the particle.

Note that the first term in the force expression is
\[ \vec{E} \vec{\alpha}_p = m_0 \gamma_p c^2 \frac{\vec{a}_p}{c^2} = m_0 \gamma_p \vec{a}_p, \]
the relativistic equivalent of \( \vec{F} = m\vec{a} \). We can also show that the force can be written as
\[ \vec{F} = E\gamma_p^2 \left[ \vec{\alpha}_p + \vec{\beta}_p \times (\vec{\beta}_p \times \vec{\alpha}_p) \right]. \]

One thing to note about this form is that cross products naturally arise whenever we transform forces from one frame to another.

Since you are actually reading the appendix, you might be interested in the details of the derivation rather than just the results.

\[
\vec{F} = \frac{d\vec{p}}{dt} = \frac{d\vec{p}\gamma}{dc} = E_0 \frac{d\vec{\gamma}}{dc} \vec{\beta}_p = E_0 \gamma_p \frac{d\vec{\beta}_p}{dc} + E_0 \vec{\beta}_p \frac{d\gamma_p}{dc}
\]
\[
= E\vec{\alpha}_p + E_0 \vec{\beta}_p \frac{d}{dc} \left( 1 - \vec{\beta}_p \cdot \vec{\beta}_p \right)^{-1/2}
\]
\[
= E\vec{\alpha}_p + E_0 \vec{\beta}_p \left( -\frac{1}{2} \left( 1 - \vec{\beta}_p \cdot \vec{\beta}_p \right)^{-3/2} \right) \left[ \vec{\beta}_p \cdot \vec{\beta}_p - \left( \frac{d}{dc} \vec{\beta}_p \right) \cdot \vec{\beta}_p \right]
\]
\[
= E\vec{\alpha}_p + E_0 \vec{\beta}_p \left( -\frac{1}{2} \gamma_p^3 \left[ -2\vec{\beta}_p \cdot \vec{\alpha}_p \right] \right)
\]
\[
= E\vec{\alpha}_p + E_0 \vec{\beta}_p \gamma_p^2 \left[ \vec{\beta}_p \cdot \vec{\alpha}_p \right]
\]
\[
= E\vec{\alpha}_p + E_0 \vec{\beta}_p \gamma_p^2 \left[ \vec{\beta}_p \times \vec{\alpha}_p \right]
\]

Now we make use of the “BAC – CAB” vector identity \( \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \):
\[
\vec{F} = E\vec{\alpha}_p + E\gamma_p^2 \vec{\beta}_p \times (\vec{\beta}_p \times \vec{\alpha}_p) + E\gamma_p^2 \vec{\beta}_p \vec{\alpha}_p
\]
\[
= E\vec{\alpha}_p + E\gamma_p^2 \vec{\beta}_p \times (\vec{\beta}_p \times \vec{\alpha}_p) + E\gamma_p^2 (1 - \vec{\beta}_p^2) \vec{\alpha}_p \quad \text{since} \quad \gamma_p^2 (1 - \vec{\beta}_p^2) = 1 \Rightarrow \gamma_p^2 \vec{\beta}_p^2 = \gamma_p^2 - 1
\]
\[
= E\gamma_p^2 \left[ \vec{\alpha}_p + \vec{\beta}_p \times (\vec{\beta}_p \times \vec{\alpha}_p) \right]
\]

4) The frequency four-vector for light:
\[
f = f \left[ \begin{array}{c} 1 \\ \hat{u} \end{array} \right]
\]
where \( f \) is the frequency in Hz and \( \hat{u} \) is a unit vector that points in the direction of the light’s travel. This relationship is just a special case of the energy-momentum four-vector with the photon energy given by Planck’s relationship \( E = hf \) where \( h \) is Planck’s constant. It also makes use of the energy-momentum relationship with \( E_0 = 0 \) for a massless particle: \( E = pc \).

### B.3 Lorentz Transformation

Let’s assume that a reference frame \( S' \) moves at a velocity \( \vec{\beta} \) with respect to frame \( S \). In the usual fashion, we let:

\[
\vec{\beta} = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}.
\]

Furthermore we stipulate that we synchronize measurements in the two reference frames by demanding that the origins coincide at \( ct = ct' = 0 \) and that the coordinate axes point in the same directions in space. Given this, we can find the space-time coordinates of any point in \( S' \) if we know then in \( S \) by using the relationships:

\[
\begin{align*}
ct' &= \gamma ct - \beta_x \gamma x - \beta_y \gamma y - \beta_z \gamma z \\
x' &= x - \beta_x \gamma x - \beta_x \gamma c t = ct \\
y' &= y - \beta_y \gamma y - \beta_y \gamma c t = ct \\
z' &= z - \beta_z \gamma z - \beta_z \gamma c t = ct
\end{align*}
\]

\[ (B.1) \]

where \( \kappa = \frac{\gamma^2}{\gamma + 1} = \frac{\gamma - 1}{\beta^2} \).

These transformations get a bit unwieldy in this form, so we usually write them as a matrix relationship:

\[
L(\vec{\beta}) = \begin{bmatrix}
\gamma & -\gamma \beta_x & -\gamma \beta_y & -\gamma \beta_z \\
-\gamma \beta_x & 1 + \kappa \beta_x^2 & \kappa \beta_x \beta_y & \kappa \beta_x \beta_z \\
-\gamma \beta_y & \kappa \beta_x \beta_y & 1 + \kappa \beta_y^2 & \kappa \beta_y \beta_z \\
-\gamma \beta_z & \kappa \beta_x \beta_z & \kappa \beta_y \beta_z & 1 + \kappa \beta_z^2
\end{bmatrix}
\]

\[
r' = L(\vec{\beta})r
\]

Even this becomes a bit tedious to write (or typeset!), so I prefer to use “dyadic notation” which makes use of the following:
\[
\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}, \quad \vec{A}^T = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix}, \quad \vec{A}^T \vec{B} = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = A_x B_x + A_y B_y + A_z B_z = \vec{A} \cdot \vec{B}
\]

\[
\vec{B} \vec{A}^T = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \begin{bmatrix} A_x & A_y & A_z \\ A_x & A_y & A_z \end{bmatrix} = \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Using this notation, the Lorentz transformation matrix reduces to the form:

(B.2 Lorentz transformation)

\[
L(\beta) = \begin{bmatrix} \gamma & -\beta \vec{\beta}^T \\ -\beta \vec{\beta} & I + \kappa \vec{\beta} \vec{\beta}^T \end{bmatrix}.
\]

Now after all this notational sleight of hand, let’s see that we get the same result as in Eq. (B.1) above:

\[
L(\beta)_1 = \begin{bmatrix} \gamma & -\beta \vec{\beta}^T \\ -\beta \vec{\beta} & I + \kappa \vec{\beta} \vec{\beta}^T \end{bmatrix} \begin{bmatrix} ct \\ \vec{r} \end{bmatrix} = \begin{bmatrix} \gamma (ct - \vec{\beta} \cdot \vec{r}) \\ -\gamma \beta ct + \vec{\beta} \cdot \vec{r} + \kappa \beta \vec{\beta} \cdot \vec{r} \end{bmatrix} = \vec{r}'.
\]

A little inspection should convince you that this is really equivalent to Eq. (B.1).

If we want to transform the primed coordinates back to the unprimed frame, then we can use the inverse transformation matrix. We could go through a lot of algebra to find the inverse, but an easier way is to make use of a physical argument. First note that \(S\) moves with respect to \(S'\) with a velocity \(-\vec{\beta}\). Since there is nothing special about one reference frame or another, to go from \(S'\) to \(S\) requires a regular Lorentz transformation with \(\vec{\beta}\) replaced by \(-\vec{\beta}\).

(B.3 Inverse transformation)

\[
L^{-1}(\vec{\beta}) = L(-\vec{\beta}) = \begin{bmatrix} \gamma & +\beta \vec{\beta} \\ +\beta \vec{\beta} & I + \kappa \beta \vec{\beta} \vec{\beta}^T \end{bmatrix}
\]

where \(S'\) moves with respect \(S\) with velocity \(\vec{\beta}\).

**B.4 Invariants**

From our experience with classical physics, we know there some physical quantities that have different numerical values in different reference frames. Such quantities are velocity and kinetic energy. Other quantities have the same value in any reference frame. These include time, length, and mass. Quantities that have the same value in all frames are called “invariants.” But we have found that classical invariants such as mass and length are not relativistic invariants. Among relativistic invariants are rest energy \((E_0 = m_0 c^2)\) and the speed of light.
It turns out there is a convenient way to find many different relativistic invariants. This method involves a matrix called the “metric tensor.” In flat space, which is all we care about unless we’re doing general relativity, the metric tensor is:

\[
\mathbf{g} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix} \equiv \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

The metric tensor is useful because of an identity that we will prove at the end of this section:

\[
\mathbf{L}(\beta) \mathbf{g} \mathbf{L}(\beta) = \mathbf{g}.
\]

To see why this is useful, we first need to define the dot product of two four-vectors:

\[
\mathbf{a} \cdot \mathbf{b} \equiv a^\top \mathbf{g} b.
\]

let’s evaluate the dot product of these four-vectors vectors in terms of components:

\[
\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix}
a_0 & a_x & a_y & a_z \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix} \begin{bmatrix}
b_0 \\
b_x \\
b_y \\
b_z \\
\end{bmatrix} = a_0 b_0 - a_x b_x - a_y b_y - a_z b_z \\
\]

or

\[
\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix}
a_0 & \bar{a}^\top \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
b_0 \\
b_x \\
b_y \\
b_z \\
\end{bmatrix} = a_0 b_0 - \bar{a} \cdot \bar{b}
\]

where \(\bar{a} \cdot \bar{b}\) is the usual dot product in three-dimensions.

Now let’s evaluate the dot product in a new reference frame \(S'\):

\[
\mathbf{a}' \cdot \mathbf{b}' = a'^\top \mathbf{g} b' = [\mathbf{L}(\bar{\beta}) \mathbf{a}]^\top \mathbf{g} [\mathbf{L}(\bar{\beta}) \mathbf{b}]
\]

\[
= a'^\top \mathbf{L}'(\bar{\beta}) \mathbf{g} \mathbf{L}(\bar{\beta}) \mathbf{b} \quad \text{since} \quad (\mathbf{AB})^\top = \mathbf{A}^\top \mathbf{B}^\top
\]

\[
= a'^\top \mathbf{L}(\bar{\beta}) \mathbf{g} \mathbf{L}(\bar{\beta}) \mathbf{b} \quad \text{since} \quad \mathbf{L}^\top = \mathbf{L}
\]

\[
= a'^\top \mathbf{g} \mathbf{b} = \mathbf{a} \cdot \mathbf{b}
\]

Thus the dot product is a relativistic invariant. Let’s see what invariants are produced by taking the dot products of a few four-vectors:
\[ \mathbf{r} \cdot \mathbf{r} = \left[ c t - \mathbf{r} \right] \left[ c t - \mathbf{r} \right] = c^2 t^2 - r^2 \] Neither distance nor time are invariant, but \( c^2 t^2 - r^2 \) is.

\[ \mathbf{p} \cdot \mathbf{p} = \left[ E \mathbf{p} - c \mathbf{p} \right] \left[ E \mathbf{p} - c \mathbf{p} \right] = E^2 - p^2 c^2 \] In the rest frame of the object, \( E = E_0 \) and \( p = 0 \), so \( E^2 - p^2 c^2 = E_0^2 \), proving the relativistic energy-momentum relationship we introduced earlier.

The one last thing we’ll do is prove Eq. (B.5):

\[
\mathbf{L}(\beta) g \mathbf{L}(\beta) = \left[ \begin{array}{cc} \gamma & -\frac{\gamma \beta^T}{\mathbf{I} + \kappa \beta \beta^T} \\ -\frac{\gamma \beta}{\mathbf{I} + \kappa \beta \beta^T} & \mathbf{I} - \frac{\gamma \beta^T}{\mathbf{I} + \kappa \beta \beta^T} \end{array} \right] \left[ \begin{array}{c} \mathbf{0}^T \\ \mathbf{0} \end{array} \right] \left[ \begin{array}{cc} \gamma & -\frac{\gamma \beta^T}{\mathbf{I} + \kappa \beta \beta^T} \\ -\frac{\gamma \beta}{\mathbf{I} + \kappa \beta \beta^T} & \mathbf{I} - \frac{\gamma \beta^T}{\mathbf{I} + \kappa \beta \beta^T} \end{array} \right]
\]

\[
= \left[ \begin{array}{cc} \gamma^2 - \gamma^2 \beta^2 & -\gamma^2 \beta^T + \gamma \beta^T + \gamma \kappa \beta^2 \beta^T \\ -\gamma^2 \beta + \gamma \beta + \gamma \kappa \beta \beta^T & \gamma^2 \beta \beta^T + \kappa \beta \beta^T - \kappa \beta \beta^T - \mathbf{1} - \kappa \beta \beta^T \end{array} \right]
\]

\[
= \left[ \begin{array}{cc} \gamma^2(1 - \beta^2) & \beta^T(-\gamma^2 + \gamma + \gamma \kappa \beta^2) \\ \beta(-\gamma^2 + \gamma + \gamma \kappa \beta^2) & \mathbf{1} + \left( \gamma^2 - 2\kappa - \kappa^2 \beta^2 \right) \beta \beta^T \end{array} \right]
\]

Let’s simplify these expressions, recalling that \( \kappa = \frac{\gamma - 1}{\beta^2} = \frac{\gamma^2}{\gamma + 1} \)

\[
\gamma^2(1 - \beta^2) = \frac{1}{1 - \beta^2(1 - \beta^2)} = 1
\]

\[
-\gamma^2 + \gamma + \gamma \kappa \beta^2 = -\gamma^2 + \gamma + \gamma \frac{\gamma - 1}{\beta^2} \beta^2 = -\gamma^2 + \gamma + \gamma(\gamma - 1) = 0
\]

\[
\gamma^2 - 2\kappa - \kappa^2 \beta^2 = \gamma^2 - \kappa \frac{\gamma - 1}{\beta^2} \beta^2 = \gamma^2 - 2\kappa - \kappa(\gamma - 1) = \gamma^2 - \kappa(\gamma + 1) = \gamma^2 - \frac{\gamma^2}{\gamma + 1}(\gamma + 1) = 0
\]

Then:

\[
\mathbf{L}(\beta) g \mathbf{L}(\beta) = \left[ \begin{array}{c} \gamma^2(1 - \beta^2) \\ \beta(-\gamma^2 + \gamma + \gamma \kappa \beta^2) \end{array} \right] \left[ \begin{array}{c} \beta^T(-\gamma^2 + \gamma + \gamma \kappa \beta^2) \\ \mathbf{1} + \left( \gamma^2 - 2\kappa - \kappa^2 \beta^2 \right) \beta \beta^T \end{array} \right]
\]

\[
= \left[ \begin{array}{c} 1 \\ \mathbf{0} \end{array} \right] = \mathbf{g}
\]
B.5 Transformation of Velocity, Force, and Acceleration

The basic method for transforming a physical quantity from one reference frame to another is to use Lorentz transformations. However, only appropriately constructed four-vectors can be transformed in this way. We can derive rules for the transformation of other quantities based on the Lorentz transformations, however.

The Transformation of Velocities

To see how this process works, let’s start with velocity. We know velocity is related to the energy momentum four-vector:

\[
p = \begin{bmatrix} E \\ \vec{p} \end{bmatrix} = \begin{bmatrix} E_0 \gamma_p \\ E_0 \gamma_p \vec{β}_p \end{bmatrix} = E_0 \gamma_p \begin{bmatrix} 1 \\ \vec{β}_p \end{bmatrix} \Rightarrow \vec{β}_p = \frac{\vec{p}c}{E}
\]

where \( \vec{β}_p \) is the velocity of a particle in the S frame.

\[
p' = \begin{bmatrix} E' \\ \vec{p}' \end{bmatrix} = \text{L}(\vec{β}) \begin{bmatrix} E \\ \vec{p} \end{bmatrix} = E_0 \gamma_p \begin{bmatrix} \gamma(1 - \vec{β} \cdot \vec{β}_p) \\ \vec{β}_p - \gamma \vec{β} + \kappa \vec{β}(\vec{β} \cdot \vec{β}_p) \end{bmatrix} = E_0 \gamma'_p \begin{bmatrix} 1 \\ \vec{β}'_p \end{bmatrix}
\]

\[
\Rightarrow \gamma'_p = \frac{E'}{E} = \gamma(1 - \vec{β} \cdot \vec{β}_p)
\]

\[
\Rightarrow \vec{β}'_p = \frac{\vec{p}'c}{E'} = \frac{\vec{β}_p - \gamma \vec{β} + \kappa \vec{β}(\vec{β} \cdot \vec{β}_p)}{\gamma(1 - \vec{β} \cdot \vec{β}_p)}
\]

The inverse transformation can be obtained by trading primed and unprimed quantities and changing the sign of \( \vec{β} \).

(B.7)

\[
\vec{β}_p = \frac{\vec{β}'_p + \gamma \vec{β} + \kappa \vec{β}(\vec{β} \cdot \vec{β}'_p)}{\gamma(1 + \vec{β} \cdot \vec{β}'_p)}
\]

As is frequently the case, the velocity transform is more easily expressed in terms of the components of the velocity parallel to the direction of \( \vec{β} \) and perpendicular to the direction of \( \vec{β} \).

We will write the components as \( \vec{β}_{p\parallel} \) and \( \vec{β}_{p\perp} \).

\[
\vec{β}'_p = \frac{\vec{β}_{p\parallel} + \vec{β}_{p\perp} - \gamma \vec{β} + \kappa \vec{β}(\vec{β}_{p\parallel})}{\gamma(1 - \vec{β}_{p\parallel})}
\]

\[
\vec{β}'_{p\parallel} = \frac{\vec{β}_{p\parallel} - \gamma \vec{β} + \kappa \vec{β}(\vec{β}_{p\parallel})}{\gamma(1 - \vec{β}_{p\parallel})} = \frac{\vec{β}_{p\parallel} - \gamma \vec{β} + \kappa \vec{β}^2 \vec{β}_{p\parallel}}{\gamma(1 - \vec{β}_{p\parallel})}
\]

\[
= \frac{\vec{β}_{p\parallel} - \gamma \vec{β} + (\gamma - 1)\vec{β}_{p\parallel}}{\gamma(1 - \vec{β}_{p\parallel})} = \frac{\vec{β}_{p\parallel} - \vec{β}}{1 - \vec{β}_{p\parallel}}
\]
\[ \beta'_{\parallel} = \frac{\beta_{\parallel} - \beta}{1 - \beta \beta_{\parallel}} \]

\[ \beta'_{\perp} = \frac{\beta_{\perp}}{\gamma(1 - \beta \beta_{\parallel})} \]

The Transformation of Forces

With forces, we may just form the force four-vector and transform it directly.

\[ f = \gamma_p \left[ \vec{F} \cdot \vec{\beta}_p \right] = \left[ \vec{F} \cdot \vec{pc} \right]. \]

Applying a Lorentz transformation to this, we obtain:

\[ f' = L(\vec{\beta}) \gamma_p \left[ \vec{F} \cdot \vec{\beta}_p \right] = \gamma_p \left[ \frac{\gamma \vec{F} \cdot \vec{\beta}_p - \vec{F} \cdot \vec{\beta}}{\vec{F} - \gamma \vec{\beta}(\vec{F} \cdot \vec{\beta}_p) + \kappa \vec{\beta}(\vec{\beta} \cdot \vec{F})} \right] = \gamma'_p \left[ \vec{F}' \cdot \vec{\beta}'_p \right] \]

\[ \vec{F}' = \frac{\gamma_p}{\gamma'_p} \left[ \vec{F} - \gamma \vec{\beta}(\vec{F} \cdot \vec{\beta}_p) + \kappa \vec{\beta}(\vec{\beta} \cdot \vec{F}) \right] \]

\[ \vec{F}' = \frac{1}{\gamma(1 - \beta \cdot \vec{\beta}_p)} \left[ \vec{F} - \gamma \vec{\beta}(\vec{F} \cdot \vec{\beta}_p) + \kappa \vec{\beta}(\vec{\beta} \cdot \vec{F}) \right] \]

We used the result \( \gamma'_p = \gamma_p \gamma(1 - \vec{\beta} \cdot \vec{\beta}_p) \) in the last step.

As with velocities, we often break the force into components parallel and perpendicular to \( \vec{\beta} \):

\[ \vec{F}' = \frac{1}{\gamma(1 - \beta \cdot \vec{\beta}_p)} \left[ \vec{F}_1 + \vec{F}_\perp - \gamma \vec{\beta}(\vec{F} \cdot \vec{\beta}_p) + \kappa \beta^2 \vec{F}_1 \right] \]

\[ \vec{F}'_\parallel = \frac{1}{\gamma(1 - \beta \cdot \vec{\beta}_p)} \left[ \vec{F}_1 - \gamma \vec{\beta}(\vec{F} \cdot \vec{\beta}_p) + (\gamma - 1) \vec{F}_1 \right] \]

\[ \vec{F}'_\perp = \frac{1}{1 - \beta \cdot \vec{\beta}_p} \left[ \vec{F}_1 - \vec{\beta}(\vec{F} \cdot \vec{\beta}_p) \right] \]

\[ \vec{F}' = \frac{\vec{F}_1}{\gamma(1 - \beta \cdot \vec{\beta}_p)} \]

(B.10)
The Transformation of Accelerations

Accelerations are a little more complicated, as they are not related to forces in as simple a way as they are in Newtonian physics. We showed in Sect. B.2 that:

\[
\vec{F} = E_p \vec{\alpha}_p + E_p \vec{\beta}_p \gamma^2_p (\vec{\beta}_p \cdot \vec{\alpha}_p)
\]

\[
= E_0 \gamma_p \vec{\alpha}_p + E_0 \vec{\beta}_p \gamma^3_p (\vec{\beta}_p \cdot \vec{\alpha}_p)
\]

where the \( p \) subscripts have been included to emphasize that the energy, velocity, and acceleration are all those of the particle experiencing the force. We can also write an expression for the power:

\[
P = \frac{dE}{dt} = c \frac{dE}{dc} = cE_0 \gamma_p \frac{d\gamma_p}{dc} = cE_0 \gamma^3_p (\vec{\beta}_p \cdot \vec{\alpha}_p)
\]

\[
P = \vec{F} \cdot \vec{v}_p \quad \text{from your mechanics course}
\]

\[
\frac{P}{c} = \vec{F} \cdot \vec{\beta}_p = E_0 \gamma^3_p (\vec{\beta}_p \cdot \vec{\alpha}_p)
\]

\[
\vec{F} = E_0 \gamma_p \vec{\alpha}_p + \vec{\beta}_p (\vec{F} \cdot \vec{\beta}_p)
\]

\[
E_0 \gamma_p \vec{\alpha}_p = \vec{F} - \vec{\beta}_p (\vec{F} \cdot \vec{\beta}_p)
\]

The important relations here are:

\[\text{(B.11)}\]

\[
\vec{F} = E_0 \gamma_p \vec{\alpha}_p + E_0 \gamma^3_p \vec{\beta}_p (\vec{\beta}_p \cdot \vec{\alpha}_p)
\]

\[
E_0 \gamma_p \vec{\alpha}_p = \vec{F} - \vec{\beta}_p (\vec{F} \cdot \vec{\beta}_p)
\]

If we know the acceleration, we can find the corresponding force. We can then transform the force as above. Finally, we can find the new acceleration in terms of the new force. The general equations are rather complicated, but we can simplify the expressions by looking at the components parallel to \( \vec{\beta} \) and perpendicular to \( \vec{\beta} \) as before.

\[
\vec{F} = E_0 \gamma (\vec{\alpha}_{p\parallel} + \vec{\alpha}_{p\perp}) + E_0 \gamma^3_p \vec{\beta}_p (\vec{\beta}_p \cdot \vec{\alpha}_{p\parallel})
\]

\[
\vec{F}_\parallel = E_0 \gamma_p \vec{\alpha}_{p\parallel} + E_0 \gamma^3_p \vec{\beta}_p (\vec{\beta}_p \cdot \vec{\alpha}_{p\parallel})
\]

\[
= E_0 \gamma_p \vec{\alpha}_{p\parallel} + E_0 \gamma_p \vec{\beta}_p \vec{\alpha}_{p\parallel}
\]

\[
= E_0 \gamma_p (1 + \gamma^2_p \vec{\beta}_p \vec{\alpha}_{p\parallel})
\]

\[
= E_0 \gamma_p \vec{\alpha}_{p\parallel} \quad \text{as} \quad \beta^2_p \gamma^2_p = \gamma^2 - 1
\]

\[
\vec{F}_\perp = E_0 \gamma_p \vec{\alpha}_{p\perp}
\]
In summary:

\[ \vec{F}_1 = E_o \gamma \vec{\alpha}_p \parallel \]
\[ \vec{F}_\perp = E_o \gamma \vec{\alpha}_p \perp \]

(B.12)

\section*{B.6 Transformation of Threads}

Since threads are massless and travel at the speed of light, the head or tail of a thread is much like a photon in the way it transforms from one reference frame to another. So let's begin by seeing how the frequency four-vector of light transforms:

\[ f' = L(\vec{\beta})f = f \left[ \gamma, -\gamma \vec{\beta}^T, \frac{1}{\gamma(1 - \vec{\beta} \cdot \hat{u})} \right] \left[ \gamma(1 - \vec{\beta} \cdot \hat{u}) \right] = f' \left[ \frac{1}{\gamma(1 - \vec{\beta} \cdot \hat{u})} \right] \]

We see that \( f' = f \gamma(1 - \vec{\beta} \cdot \hat{u}) \) and that the unit vector pointing in the direction of the photon motion transforms as:

(B.13) \[ \hat{u}' = \frac{\hat{u} - \gamma \vec{\beta} + \kappa \vec{\beta}(\vec{\beta} \cdot \hat{u})}{\gamma(1 - \vec{\beta} \cdot \hat{u})} \] or

(B.14) \[ \left[ \frac{1}{\hat{u}'} \right] = \frac{1}{\gamma(1 - \vec{\beta} \cdot \hat{u})} L(\vec{\beta}) \left[ \frac{1}{\hat{u}} \right]. \]

We could also use the inverse transformation to go the other way in exactly the same fashion. This gives:

(B.15) \[ \left[ \frac{1}{\hat{u}} \right] = \frac{1}{\gamma(1 + \vec{\beta} \cdot \hat{u}')} L(-\vec{\beta}) \left[ \frac{1}{\hat{u}'} \right], \quad \hat{u} = \frac{\hat{u}' + \gamma \vec{\beta} + \kappa \vec{\beta}(\vec{\beta} \cdot \hat{u}')}{{\gamma}(1 + \vec{\beta} \cdot \hat{u}'}). \]

Note that \( \left[ \frac{1}{\hat{u}} \right] \) is not really a four-vector since it doesn’t transform by a simple Lorentz transformation because of the factor of \( \gamma(1 - \vec{\beta} \cdot \hat{u}) \) that appears in Eq. (B.15). We will call \( \left[ \frac{1}{\hat{u}} \right] \) the “direction vector” and use it extensively in Appendix C.

\section*{B.7 Transformation of Fields and the Field Strength Tensor}

If there are many source charges, each moving with a different velocity, the net force on a field charge is the vector sum of all the forces.
\[ \vec{F}_{\text{net}} = \sum_i [q_i \vec{E}_i + q_i \vec{v}_f \times \vec{B}_i] = q_f \sum_i \vec{E}_i + q_f \vec{v}_f \times \sum_i \vec{B}_i = q_f \vec{E}_{\text{net}} + q_f \vec{v}_f \times \vec{B}_{\text{net}} \]

So in a given reference frame the most general force on a field particle is:

\[ \vec{F}_f = q_f \vec{E}_s + q_f \vec{v}_f \times \vec{B}_s = q_f \vec{E}_s + q_f \vec{\beta}_f \times \vec{cB}_s. \]

In these equations the \( f \) refers to a field particle and the \( s \) to a source particle. We know how to transform forces from one frame to another, so in principle we can transform electric and magnetic fields. If we know the fields of a group of source charges and the velocity of a field charge, we can 1) find the force in one reference frame, 2) transform the force to another frame, 3) transform the velocity to another frame, and 4) regroup all the terms to find the new fields. The algebra involved in this process, however, is very tedious. But by making clever use of our matrix notation, we can simplify the process a great deal.

The first step is to write out the Lorentz force equation in terms of column vectors:

\[
\begin{bmatrix}
F_{f_x} \\
F_{f_y} \\
F_{f_z}
\end{bmatrix} = q_f \begin{bmatrix}
E_x + \beta_{f x} c B_z - \beta_{f y} c B_y \\
E_y + \beta_{f y} c B_x - \beta_{f z} c B_z \\
E_z + \beta_{f z} c B_y - \beta_{f x} c B_x
\end{bmatrix}.
\]

Then we can formulate the force four-vector. First, the time component is:

\[
\gamma_f \vec{F}_f \cdot \vec{\beta}_f = \gamma_f q_f \vec{E} \cdot \vec{\beta}_f + \gamma_f q_f (\vec{\beta}_f \times \vec{cB}) \cdot \vec{\beta}_f
\]

\[= \gamma_f q_f \vec{E} \cdot \vec{\beta}_f \]

The last step follows since the cross product is perpendicular to \( \vec{\beta}_f \), so the dot product of \( \vec{\beta}_f \) with it is zero. (The dot product of perpendicular vectors is zero.) This gives us:

\[
\begin{bmatrix}
\vec{F}_f \cdot \vec{\beta}_f \\
F_{f_x} \\
F_{f_y} \\
F_{f_z}
\end{bmatrix} = \gamma_f q_f \begin{bmatrix}
E_x \beta_{f x} + E_y \beta_{f y} + E_z \beta_{f z} \\
E_x + \beta_{f x} c B_z - \beta_{f y} c B_y \\
E_y + \beta_{f y} c B_x - \beta_{f z} c B_z \\
E_z + \beta_{f z} c B_y - \beta_{f x} c B_x
\end{bmatrix}.
\]

Now, we get to the trick: we write this as a matrix expression to separate the force’s field dependence from its velocity dependence.
\[ f = \gamma_j q_f \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \beta_{f_x} \\ \beta_{f_y} \\ \beta_{f_z} \end{bmatrix} \equiv q_f M \frac{p_f}{E_{0f}} \]

where
\[
M \text{ is a matrix called the “field strength tensor.”}
\]
\[
p_f \text{ is the energy-momentum four-vector of the field particle.}
\]
\[
E_{0f} \text{ is the rest energy of the field particle.}
\]

With this form, we can transform the force four-vector to another frame:
\[
f' = L(\vec{\beta})f = q_f L(\vec{\beta})M \frac{p_f}{E_{0f}}
\]
\[
= q_f L(\vec{\beta})ML(-\vec{\beta})L(\vec{\beta}) \frac{p_f}{E_{0f}} \text{ since } L(-\vec{\beta})L(\vec{\beta}) = I
\]
\[
= q_f M' \frac{p_f}{E_{0f}}
\]
\[
\Rightarrow M' = L(\vec{\beta})ML(-\vec{\beta})
\]

This last line shows us how the field strength tensor transforms into another reference frame.

Before we multiply this expression out, let’s rewrite the field strength tensor in a simpler form:
\[
M = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & \vec{E}' \\ \vec{E} & \mathbf{B} \end{bmatrix}
\]

(B.16)

where \( \mathbf{B} = \begin{bmatrix} 0 & cB_z & -cB_y \\ -cB_z & 0 & cB_x \\ cB_y & -cB_x & 0 \end{bmatrix} \).

If we let \( \vec{A} \) be any arbitrary vector, we see that:
\[
\begin{bmatrix}
0 & cB_z & -cB_y \\
-cB_z & 0 & cB_x \\
cB_y & -cB_x & 0 \\
\end{bmatrix}
\begin{bmatrix}
A_x \\
A_y \\
A_z \\
\end{bmatrix}
= 
\begin{bmatrix}
cB_z A_y - cB_y A_z \\
cB_x A_z - cB_z A_x \\
cB_y A_x - cB_x A_y \\
\end{bmatrix}
= \tilde{A} \times c\tilde{B}
\]

\[
\begin{bmatrix}
0 & cB_z & -cB_y \\
-cB_z & 0 & cB_x \\
cB_y & -cB_x & 0 \\
\end{bmatrix}
= \begin{bmatrix}
A_x cB_y - A_y cB_z \\
A_y cB_z - A_z cB_x \\
A_z cB_x - A_x cB_y \\
\end{bmatrix}
= (-\tilde{A} \times c\tilde{B})^T
\]

Now we can find \( M' \):

\[
M' = L(\tilde{\beta})ML(-\tilde{\beta}) = \begin{bmatrix}
\gamma & -\gamma\tilde{\beta}^T \\
-\gamma\tilde{\beta} & I + \kappa\tilde{\beta}\tilde{\beta}^T \\
\end{bmatrix}
\begin{bmatrix}
0 & E^T \\
E & B \\
\gamma & \gamma\tilde{\beta}^T \\
\end{bmatrix} = \begin{bmatrix}
\gamma(\tilde{E} \cdot \tilde{\beta}) & \tilde{E}^T + \kappa(\tilde{\beta} \cdot \tilde{E})\tilde{\beta}^T \\
\gamma\tilde{E} + (\gamma\tilde{\beta} \times c\tilde{B}) & \gamma\tilde{E}\tilde{\beta}^T + B + \kappa(\tilde{\beta} \times c\tilde{B})\tilde{\beta}^T \\
\end{bmatrix}
\]

\[
M'_{1,1} = \gamma^2 (\tilde{E} \cdot \tilde{\beta}) - \gamma^2 (\tilde{E} \cdot \tilde{\beta}) - \gamma^2 \tilde{\beta} \cdot (\tilde{\beta} \times c\tilde{B})
\]

\[= 0 \quad \text{as} \quad \tilde{a} \cdot (\tilde{a} \times \tilde{b}) = 0\]

\[
M'_{1,2-4} = \gamma\tilde{E}^T + \kappa\gamma(\tilde{\beta} \cdot \tilde{E})\tilde{\beta}^T - \gamma^2 (\tilde{\beta} \cdot \tilde{E})\tilde{\beta}^T + \gamma(\tilde{\beta} \times c\tilde{B})^T - \kappa\gamma(\tilde{\beta} \cdot (\tilde{\beta} \times c\tilde{B}))\tilde{\beta}^T
\]

\[= \gamma\tilde{E}^T - \kappa(\tilde{\beta} \cdot \tilde{E})\tilde{\beta}^T + \gamma(\tilde{\beta} \times c\tilde{B})^T \quad \text{as} \quad \tilde{a} \cdot (\tilde{a} \times \tilde{b}) = 0 \quad \text{and} \quad \kappa - \gamma = -\frac{\kappa}{\gamma}\]

\[
M'_{2-4,1} = -\gamma^2 (\tilde{\beta} \cdot \tilde{E})\tilde{\beta} + \gamma\tilde{E} + \gamma(\tilde{\beta} \times c\tilde{B}) + \kappa\gamma(\tilde{\beta} \cdot \tilde{E})\tilde{\beta} + \kappa\gamma(\tilde{\beta} \cdot (\tilde{\beta} \times c\tilde{B}))\tilde{\beta}
\]

\[= \gamma\tilde{E} - \kappa(\tilde{\beta} \cdot \tilde{E})\tilde{\beta} + \gamma(\tilde{\beta} \times c\tilde{B})\]

\[
M'_{2-4,2-4} = -\gamma\tilde{E}^T - \kappa\gamma(\tilde{\beta} \cdot \tilde{E})\tilde{\beta}\tilde{\beta}^T + \gamma\tilde{E} \tilde{\beta}^T + B + \kappa(\tilde{\beta} \times c\tilde{B})\tilde{\beta}^T + \kappa\gamma(\tilde{\beta} \cdot \tilde{E})\tilde{\beta}\tilde{\beta}^T - \kappa\gamma(\tilde{\beta} \cdot (\tilde{\beta} \times c\tilde{B}))\tilde{\beta}\tilde{\beta}^T
\]

\[= B + \kappa(\tilde{\beta} \times c\tilde{B})\tilde{\beta}^T - \kappa\tilde{\beta}(\tilde{\beta} \times c\tilde{B})^T + \gamma\tilde{E} \tilde{\beta}^T - \gamma\tilde{E}^T
\]

\[
M' = \begin{bmatrix}
0 & \gamma\tilde{E}^T + (\tilde{\beta} \times c\tilde{B})^T - \kappa(\tilde{\beta} \cdot \tilde{E})\tilde{\beta}^T \\
\gamma\tilde{E} + (\tilde{\beta} \times c\tilde{B}) - \kappa(\tilde{\beta} \cdot \tilde{E})\tilde{\beta} & B + \kappa(\tilde{\beta} \times c\tilde{B})\tilde{\beta}^T - \kappa\tilde{\beta}(\tilde{\beta} \times c\tilde{B})^T + \gamma\tilde{E} \tilde{\beta}^T - \gamma\tilde{E}^T
\end{bmatrix}
\]

From this matrix, we can read off the fields in the \( \mathbf{R}' \) frame.
\[ \tilde{E}' = \gamma(E + \tilde{\beta} \times c\tilde{B}) - \kappa(\tilde{\beta} \cdot \tilde{E})\tilde{\beta} \]
\[ c\beta' = M'_{3,4} = cB_x + \kappa(\tilde{\beta} \times c\tilde{B})_y \beta_z - \kappa(\tilde{\beta} \times c\tilde{B})_z \beta_y + \gamma E_y \beta_z - \gamma E_z \beta_y \]
\[ = cB_x + \kappa[(\tilde{\beta} \times c\tilde{B})_y]_z + \gamma(\tilde{\beta} \times \tilde{E})_x \]
\[ \Rightarrow c\tilde{B}' = c\tilde{B} - \kappa\tilde{\beta} \times (\tilde{\beta} \times c\tilde{B}) - \gamma(\tilde{\beta} \times \tilde{E}) \quad \tilde{a} \times \tilde{b} = -\tilde{b} \times \tilde{a} \]
\[ = c\tilde{B} - \kappa\tilde{\beta}(\tilde{\beta} \cdot c\tilde{B}) + \kappa c\tilde{B} \beta^2 - \gamma(\tilde{\beta} \times \tilde{E}) \quad "BAC" \text{ minus } "CAB" \text{ Rule} \]
\[ = c\tilde{B} - \kappa\tilde{\beta}(\tilde{\beta} \cdot c\tilde{B}) + (\gamma - 1)c\tilde{B} - \gamma(\tilde{\beta} \times \tilde{E}) \]
\[ = \gamma(c\tilde{B} - \tilde{\beta} \times \tilde{E}) - \kappa\tilde{\beta}(\tilde{\beta} \cdot c\tilde{B}) \]

In summary,

**Field Transformation equations**

(B.17)
\[ \tilde{E}' = \gamma(E + \tilde{\beta} \times c\tilde{B}) - \kappa(\tilde{\beta} \cdot \tilde{E})\tilde{\beta} \]
\[ c\beta' = \gamma(c\tilde{B} - \tilde{\beta} \times \tilde{E}) - \kappa(\tilde{\beta} \cdot c\tilde{B})\tilde{\beta} \]

where

\( R' \) moves with velocity \( \tilde{\beta} \) with respect to \( R \).

\( \tilde{E} \) and \( \tilde{B} \) are the fields in \( R \).

\( \tilde{E}' \) and \( \tilde{B}' \) are the fields in \( R' \).

\( \tilde{F}_f = q_f \tilde{E} + q_f \tilde{\beta} \times c\tilde{B} \) in \( R \).

\( \tilde{F}_f' = q_f \tilde{E}' + q_f \tilde{\beta}' \times c\tilde{B}' \) in \( R' \).

The field strength tensor is defined a little differently in many texts. In terms of \( M \) and the metric tensor \( \mathbf{g} \), it is often written as:

\[ \mathbf{T} = M \mathbf{g} \]

\[ \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -\tilde{E}^T \\ \tilde{E} & -\mathbf{B} \end{bmatrix} \]
In this notation the force four-vector is:

\[ f = q_f M \frac{p_f}{E_{0f}} = q_f M g g \frac{p_f}{E_{0f}} = q_f T g \frac{p_f}{E_{0f}} \quad \text{as } gg = 1 \]

\[ f' = q_f L(\tilde{\beta}) T g \frac{p_f}{E_{0f}} = q_f L(\tilde{\beta}) TL(\tilde{\beta}) g L(\tilde{\beta}) \frac{p_f}{E_{0f}} \quad \text{as } L g L = g \]

\[ f' = q_f T' g \frac{p_f}{E_{0f}} \Rightarrow T' = L(\tilde{\beta}) TL(\tilde{\beta}) \]

The same conclusions follow in a similar fashion.