Physics 571 Lecture #3

1 Stability of Laser Cavities

In what follows, we apply the ABCD matrix formulation to a laser cavity. The basic elements of a laser cavity include an amplifying medium and mirrors to provide feedback. Presumably, at least one of the end mirrors is partially transmitting so that energy is continuously extracted from the cavity. Here, we dispense with the amplifying medium and concentrate our attention on the optics providing the feedback. As might be expected, the mirrors must be carefully aligned or successive reflections might cause rays to ‘walk’ continuously away from the optical axis, so that they eventually leave the cavity out the side. If a simple cavity is formed with two flat mirrors that are perfectly aligned parallel to each other, one might suppose that the mirrors would provide ideal feedback. However, all rays except for those that are perfectly aligned to the mirror surface normals eventually wander out of the side of the cavity as illustrated in Fig. 1. Such a cavity is said to be unstable. We would like to do a better job of trapping the light in the cavity.

Figure 1: A ray of light bouncing between two parallel flat mirrors.

To improve the situation, a cavity can be constructed with concave end mirrors to help confine the beams within the cavity. Even so, one must choose carefully the curvature of the mirrors and their separation $L$. If this is not done correctly, the curved mirrors can ‘overcompensate’ for the tendency of the rays to wander out of the cavity and thus aggravate the problem. Such an unstable scenario is depicted in Fig. 2.

Figure 2: A ray of light bouncing between two curved mirrors in an unstable configuration.

Figure 3 depicts a cavity made with curved mirrors where the separation $L$ is chosen appropriately to make the cavity stable. Although a ray, as it makes successive bounces, can strike the end mirrors at a variety of points, the curvature of the mirrors keeps the ‘trajectories’ contained within a narrow region so that they do not escape out the sides of the cavity.

There are many ways to make a stable laser cavity. For example, a stable cavity can be made using a lens between two flat end mirrors as shown in Fig. 4. Any combination of lenses (perhaps more than one) and curved mirrors can be used to create stable cavity configurations. ‘Ring cavities’
can also be made to be stable where in no place do the rays retro-reflect from a mirror but circulate through a series of elements like cars going around a racetrack.

As far as stability of the cavity goes, we are interested in knowing what a ray does after making many round trips in the cavity. To find the effect of propagation through many round trips, we multiply the round-trip ABCD matrix together \( N \) times, where \( N \) is the number of round trips that we wish to consider. We can then examine what happens to an arbitrary ray after making \( N \) round trips in the cavity as follows:

\[
\begin{bmatrix}
y_2 \\
\theta_2
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} y_1 \\
\theta_1
\end{bmatrix}
\]

You may be concerned about the difficulty of evaluating an ABCD matrix raised to the \( N^{th} \), especially since \( N \) may be very large, perhaps even infinity. However, we can use a neat trick to accomplish this otherwise daunting task.
We use Silvester’s theorem, proved in the appendix below, which states:

\[
\begin{vmatrix} \sin \theta & 1 \\ -1 & \sin \theta \end{vmatrix} = 1,
\]

then

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin \theta} \begin{bmatrix} A \sin N\theta - \sin(N-1)\theta & B \sin N\theta \\ C \sin N\theta & D \sin N\theta - \sin(N-1)\theta \end{bmatrix}, \text{ where}
\]

\[
\cos \theta = \frac{1}{2} (A + D).
\]

(4)

It turns out that Eq. 4 is satisfied if the ABCD matrix is for a ray starting and ending in the same refractive index, which is guaranteed for any round trip. We therefore can employ Silvester’s theorem for any \(N\) that we might choose, including very large integers.

We would like the elements of Eq. 5 to remain finite as \(N\) becomes very large. If this is the case, then we know that a ray remains trapped within the cavity and stays reasonably close to the optical axis. Since \(N\) only appears within the argument of a sine function, which is always bounded between -1 and 1 for real arguments, it might seem that the elements of Eq. 5 always remain finite as \(N\) approaches infinity. However, it turns out that \(\theta\) can become imaginary depending on the outcome of Eq. 6, in which case the sine becomes a hyperbolic sine, which can ‘blow up’ as \(N\) becomes large. In the end, the condition for cavity stability is that a real \(\theta\) must exist for Eq. 6, or in other words we need

\[-1 \leq \frac{1}{2} (A + D) \leq 1. \quad \text{(condition for a stable cavity)} \]

(7)

It is left as an exercise to apply this condition to Eq. 1 and Eq. 2 as well as other cavity geometries to find the necessary relationships between the various element curvatures and spacing in order to achieve cavity stability.

2 Appendix

Silvesters theorem may be proved by induction. When \(N = 1\), the equation is seen to be correct. Next we assume that the theorem holds for arbitrary \(N\) and check to see if it holds for \(N + 1:\)

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{N+1} = \frac{1}{\sin \theta} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A \sin N\theta - \sin(N-1)\theta & B \sin N\theta \\ C \sin N\theta & D \sin N\theta - \sin(N-1)\theta \end{bmatrix}
\]

\[
= \frac{1}{\sin \theta} \begin{bmatrix} (A^2 + BC) \sin N\theta - A \sin(N-1)\theta & (AB + BD) \sin N\theta - B \sin(N-1)\theta \\ (AC + CD) \sin N\theta - C \sin(N-1)\theta & (D^2 + BC) \sin N\theta - D \sin(N-1)\theta \end{bmatrix}
\]

\[
= \frac{1}{\sin \theta} \begin{bmatrix} (A^2 + AD - 1) \sin N\theta - A \sin(N-1)\theta & B[(A + D) \sin N\theta - \sin(N-1)\theta] \\ C[(A + D) \sin N\theta - \sin(N-1)\theta] & (D^2 + AD - 1) \sin N\theta - D \sin(N-1)\theta \end{bmatrix}.
\]
We used $AD - BC = 1$, namely Eq. 4, in simplifying the diagonal elements. Further rearrangement gives

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{N+1} = \frac{1}{\sin \theta} \begin{bmatrix} A[(A + D) \sin N\theta - \sin(N - 1)\theta] - \sin N\theta & B[(A + D) \sin N\theta - \sin(N - 1)\theta] \\ C[(A + D) \sin N\theta - \sin(N - 1)\theta] & D[(A + D) \sin N\theta - \sin(N - 1)\theta] - \sin N\theta \end{bmatrix}.$$ 

In each matrix element, the expression

$$(A + D) \sin N\theta = 2 \cos \theta \sin N\theta = \sin(N + 1)\theta + \sin(N - 1)\theta$$

occurs, which we have rearranged using $\cos \theta = \frac{1}{2} (A + D)$. The result is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{N+1} = \frac{1}{\sin \theta} \begin{bmatrix} A \sin(N + 1)\theta - \sin N\theta & B \sin(N + 1)\theta \\ C \sin(N + 1)\theta & D \sin(N + 1)\theta - \sin N\theta \end{bmatrix},$$

which completes the proof.