Physics 513 Homework

Given 1/4
1. Given that \( \vec{L} = \vec{r} \times \vec{p} \), show that \( [L_x, L_y] = i\hbar L_z \).

Given 1/6
2. Find the eigenvalues and eigenvectors of \( \sigma_x \) and \( \sigma_y \).
3. Show that \( S^2 = \frac{3\hbar^2}{4} \mathbf{I} \).
4. (a) Expand the state \( |\ell m_{s} m_{s}\rangle = |2 1 \frac{1}{2} - \frac{1}{2}\rangle \) in terms of \( |\ell s m_j\rangle \) basis functions.
   (b) Expand the state \( |\ell s m_j\rangle = |2 \frac{15}{2} \frac{5}{2} \rangle \) in terms of \( |\ell m_{s} m_{s}\rangle \) basis functions.
   (c) Show in each case that the sum of the squares of the coefficients = 1.

Given 1/9
5. Use Clebsch-Gordan coefficients to find the expansions of \( \chi_1^1 \), \( \chi_1^0 \), and \( \chi_1^{-1} \) in terms \( \alpha_1 \), \( \alpha_2 \), \( \beta_1 \), and \( \beta_2 \).
6. By explicit evaluation, find the results of \( \hat{S}_{1x} \), \( \hat{S}_{1y} \), and \( \hat{S}_{1z} \) operating on \( \alpha_1 \), and \( \beta_1 \).
7. (a) Find \( \hat{S}^2 \alpha_1 \beta_2 \) and \( \hat{S}^2 \alpha_2 \beta_1 \) by explicit evaluation.
   (b) Show \( \hat{S}^2 \frac{1}{\sqrt{2}} (\alpha_1 \beta_2 - \alpha_2 \beta_1) \) has the eigenvalue 0.
   (c) Show \( \hat{S}^2 \frac{1}{\sqrt{2}} (\alpha_1 \beta_2 + \alpha_2 \beta_1) \) has the eigenvalue \( 2\hbar^2 \).
   (d) Find the result of \( \hat{S}_{1z} \) operating on each of the wavefunctions of parts (b) and (c).

Given 1/11
8. Are the following good states for p+p? p+n? (See the top of p. 179.)
   \( 3S_1, 1P_1, 3D_2, 3P_0, 3F_0 \)

Given 1/13
9. (a) \( ^{16}\text{O} \) has isospin 0. What are the possible isospins of \( ^{17}\text{O} \) and \( ^{18}\text{O} \)?
   What are the possible isospins of the \( ^{17}\text{O}^* \) in the following reactions?
   (b) \( ^{17}\text{O}(\alpha, \alpha')^{17}\text{O}^* \)
   (c) \( ^{17}\text{O}(d, d')^{17}\text{O}^* \)
   (d) \( ^{16}\text{O}(d, p)^{17}\text{O}^* \)
   (e) \( ^{18}\text{O}(^{3}\text{He}, \alpha)^{17}\text{O}^* \)
10. Find the ratio of cross sections for the reactions \( p + p \rightarrow d + \pi^+ \) and \( p + n \rightarrow d + \pi^0 \).
11. Show that \( \tau_+(1) \tau_-(2) + \tau_-(1) \tau_+(2) = \frac{1}{2} [\tau_1(1) \tau_1(2) + \tau_2(1) \tau_2(2)] \).

Given 1/18
12. Using a well depth of \( V_0 = 50 \text{ MeV} \) and \( \mu = 0.707 \text{ fm}^{-1} \) plot all the potentials listed on p. 209 to compare the shapes.
Given 1/20

13. We want to prove two rather long identities, but we’ll do it a step at a time to make life easier. Each identity involves spin operators for particles 1 and 2. Remember that operators in different spin spaces commute.

Here are some things we’ve proven before: $\sigma_{1x}\sigma_{1y} = i\sigma_{1z}$, $\sigma_{1y}\sigma_{1x} = -i\sigma_{1z}$, $\sigma_{1x}\sigma_{1x} = 1$ et. cyc.

Given:
$S = \hbar(\sigma_1 + \sigma_2)/2$ is the total spin operator
$H = (S \cdot \vec{r})/r$ is the Helicity operator
$S_{12} = 3(\sigma_1 \cdot \vec{r})(\sigma_2 \cdot \vec{r})/r^2 - (\sigma_1 \cdot \sigma_2)$ is the tensor operator

(a) Prove that $\hbar^2 S_{12} = 6H^2 - 2S^2$. You will probably find it helpful to expand $(\sigma_1 \cdot \vec{r})(\sigma_2 \cdot \vec{r})$ and $(S \cdot \vec{r})^2$ and combine the spin matrices. Note: You don’t need to write all the terms – feel free to use … liberally. (This is the first proof. The rest is the second proof.)

(b) We may let:
$A_1 = (\sigma_1 \cdot \vec{r})/r$ an operator in the space of particle 1
$A_2 = (\sigma_2 \cdot \vec{r})/r$ an operator in the space of particle 2
$A_{12} = (\sigma_1 \cdot \sigma_2)$ an operator with pieces in both spin states.

Then $S_{12} = 3A_1A_2 - A_{12}$.

Show that $S_{12}^2 = 9A_1^2A_2^2 - 3A_{12}A_1A_2 - 3A_1A_2A_{12} + A_{12}^2$
(That one’s trivial, but it’s a reminder that operator order is important!)

(c) To evaluate $A_1A_2A_{12}$, you’ll need to find terms such as these:

<table>
<thead>
<tr>
<th>First factors \ Last</th>
<th>$\sigma_{1x}\sigma_{2x}$</th>
<th>$\sigma_{1y}\sigma_{2y}$</th>
<th>$\sigma_{1z}\sigma_{2z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{1x}\sigma_{2x}$</td>
<td>1</td>
<td>$\sigma_{1z}\sigma_{2z}$</td>
<td>$\sigma_{1z}\sigma_{2z}$</td>
</tr>
<tr>
<td>$\sigma_{1x}\sigma_{2y}$</td>
<td>$-i\sigma_{2z}$</td>
<td>$i\sigma_{1x}$</td>
<td>$\sigma_{1y}\sigma_{2x}$</td>
</tr>
<tr>
<td>$\sigma_{1x}\sigma_{2z}$</td>
<td>$i\sigma_{2y}$</td>
<td>$\sigma_{1z}\sigma_{2x}$</td>
<td>$-i\sigma_{1z}$</td>
</tr>
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<td>$\sigma_{1y}\sigma_{2x}$</td>
<td>$-i\sigma_{1z}$</td>
<td>$i\sigma_{2y}$</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_{1y}\sigma_{2y}$</td>
<td>$-\sigma_{1y}\sigma_{2x}$</td>
<td>$\sigma_{1y}\sigma_{2x}$</td>
<td>$-i\sigma_{2x}$</td>
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<td>$\sigma_{1y}\sigma_{2z}$</td>
<td>$i\sigma_{1y}$</td>
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<td>$\sigma_{1z}\sigma_{2x}$</td>
<td>$\sigma_{1y}\sigma_{2z}$</td>
<td>$-i\sigma_{1x}$</td>
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<td>$\sigma_{1z}\sigma_{2y}$</td>
<td>$-\sigma_{1z}\sigma_{2x}$</td>
<td>$\sigma_{1z}\sigma_{2x}$</td>
<td>$-\sigma_{1y}\sigma_{2x}$</td>
</tr>
</tbody>
</table>

Fill in the missing terms and find all my mistakes. (They’re not intentional, but they may be there.)

(d) Show that $A_1A_2A_{12} = 1 - A_{12} + A_1A_2$
(e) To evaluate $A_{12}A_1A_2$, you’ll need to fill out the following table. Show that $A_{12}A_1A_2 = 1 - A_{12} + A_1A_2$ also.

<table>
<thead>
<tr>
<th>Last factors x First</th>
<th>$\sigma_{1x}\sigma_{2x}$</th>
<th>$\sigma_{1y}\sigma_{2y}$</th>
<th>$\sigma_{1z}\sigma_{2z}$</th>
</tr>
</thead>
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<tr>
<td>$\sigma_{1x}\sigma_{2x}$</td>
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<td>$\sigma_{1x}\sigma_{2y}$</td>
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<td>$\sigma_{1y}\sigma_{2y}$</td>
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<td>$\sigma_{1z}\sigma_{2x}$</td>
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<td>$\sigma_{1z}\sigma_{2y}$</td>
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<tr>
<td>$\sigma_{1z}\sigma_{2z}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(f) Show that $A_{12}^2 = 3 - 2A_{12}$.

(g) Show that $A_1^2 = 1$.

(h) Now combine everything to show that $S_{12}^2 = 6 + 2(\sigma_1 \cdot \sigma_2) - 2S_{12}$.

Given 1/25
14. Using Mathematica, show that the expressions

$$u(r) = N_S e^{-\frac{r}{R}}$$

$$v(r) = N_D e^{-\frac{r}{R}} \left[ 1 + 3\frac{R}{r} + 3 \left( \frac{R}{r} \right)^2 \right]$$

are asymptotic (large $r$) solutions to the differential equations (12.19 and 12.20, corrected):

$$\frac{\hbar^2}{M} \frac{d^2}{dr^2} u(r) + \left[ E - V_C(r) \right] u(r) - 8V_T(r)v(r) \tan \omega = 0$$

$$\frac{\hbar^2}{M} \frac{d^2}{dr^2} v(r) + \left[ E - \frac{6\hbar^2}{Mr^2} - V_C(r) + 2V_T(r) \right] v(r) - \sqrt{8}V_T(r)u(r) \cot \omega = 0$$

Given 1/27
15. (a) Protons with a kinetic energy of 2.17 MeV are incident on $^{27}$Al. Plot the cm scattering angle for elastic scattering versus the lab scattering angle (in degrees).

(b) Repeat the same for 500 MeV protons.

Nuclear masses can be found in the Appendix 4 of the text. It is useful to note that

$1 \text{ u} = 931.4941 \text{ MeV}$ and $m_c = 0.5109989 \text{ MeV}$. 
16. A proton-nucleus potential that has proven to be quite reliable for proton kinetic energies, $T_p$, in the range of 80-180 MeV is the optical model potential of Schwandt, et al. (Phys Rev C, 26:55 (1982)). Its general form is quite similar to the OBEP potential of Chapter 11. The general form is

$$U(r) = U_{\text{coul}}(r) - Vf_0(r) - iWf_w(r) + \frac{2}{r} \left[ V_{so} \frac{d}{dr} f_{vso}(r) + iW_{so} \frac{d}{dr} f_{wso}(r) \right] \vec{L} \cdot \vec{\sigma}$$

Where all the $f$ functions are of the Woods-Saxon form, such as:

$$f_0(r) = \frac{1}{1 + \exp\left(\frac{r - r_0}{a_0}\right)}$$

For this assignment, we will ignore all but the central, real part of the potential. We’ll even ignore the Coulomb part of the potential. The parameters we’ll need to know are:

$$V = 105.5(1 - 0.1625 \ln T_p) + 16.5(N - Z)/A \quad \text{(MeV)}$$

$$r_0 = \left[ 1.125 + 1.0 \times 10^{-3} T_p \right] A^{1/3} \quad \text{(fm)}$$

$$a_0 = 0.675 + 3.1 \times 10^{-4} T_p \quad \text{(fm)}$$

(a) Pick a stable nucleus of your choice with $A > 30$. Estimate the nuclear diameter by using the relationship $R = 1.2 \text{ fm} A^{1/3}$. Plot $Vf_0(r)$ and see how the radius compares with this. Estimate the number of partial waves you’ll need for 100 MeV protons.

(b) Start with the $\ell = 0$ partial wave. Using MATLAB, integrate out the solution to a radius past the range of the potential. At the matching radius, find the value of the phase shift, $\delta_0$. Calculate the differential cross section ($d\sigma/d\Omega$) as a function of cm scattering angle based on this partial wave alone.

(c) Repeat (b) for the next two partial waves.

(d) Find phase shifts of all the partial waves you need, and plot the full differential cross section.

Here are a couple of helps:

The radial equation is:

$$\frac{d^2}{dr^2} (rR_\ell) = \left[ \frac{\ell(\ell + 1)}{r^2} + \bar{U}(r) - k^2 \right] (rR_\ell)$$

where

$$\bar{U}(r) = \frac{2\mu}{\hbar^2} U(r) \quad \text{and} \quad k^2 = \frac{2\mu}{\hbar^2} T_{cm}$$

with $T_{cm}$ being the total kinetic energy in the cm frame.
To “integrate out” the Schrödinger Equation (there’s no regular integral involved), let \( \phi(r) = rR_\ell \) with \( \phi(0) = 0 \) and \( \phi'(0) = 1 \). The latter is acceptable because the wave function isn’t yet normalized. Then all you need to do is iterate outward:

\[
\begin{align*}
\phi(r + dr) &= \phi(r) + \phi'(r)dr \\
\phi'(r + dr) &= \phi'(r) + \phi''(r)dr
\end{align*}
\]

and \( \phi''(r) \) is obtained from the radial equation. There are more efficient methods when the first derivative is not used (Fox-Goodwin Method #7 and Numerov), but we don’t care a lot about efficiency here.

Integrate the wave function out to a radius far beyond the potential. You might check to see that the wave function for the last partial wave has settled down to “regular” oscillations at the matching radius you have chosen.

A good check is to set \( U(r) = 0 \) and be sure that your wave function matches the regular spherical Bessel function. Since MATLAB doesn’t have spherical Bessel functions, you’ll need to use the identity

\[
\begin{align*}
\tau j_\ell(k\tau) &= \frac{\pi \tau}{2k} J_{\ell+1/2}(k\tau)
\end{align*}
\]

After you’ve integrated the differential equation out to the matching radius, you’ll want to equate the function and second derivative to the spherical Bessel functions. There are two ways of doing this:

1) The traditional way is to use the asymptotic form of the combined spherical Bessel and Neumann functions (see notes) and its derivative:

\[
\begin{align*}
\phi_\ell(r) &= rR_\ell(kr) \to \frac{1}{k} \sin \left( kr - \frac{\ell\pi}{2} + \delta_\ell \right) \\
\phi'_\ell(r) &\to \cos \left( kr - \frac{\ell\pi}{2} + \delta_\ell \right)
\end{align*}
\]

A simple way to equate the function and its first derivative is to equate the ratio:

\[
\frac{k\phi_\ell}{\phi'_{\ell}} \to \tan \left( kr - \frac{\ell\pi}{2} + \delta_\ell \right)
\]

It turns out that you have to choose a matching radius that is very large (a few thousand fm) or the asymptotic form isn’t very good. But if you integrate the wave function out that far, rounding errors accumulate. These are important particularly at large angle scattering and for large \( \ell \).

Also, the choice of the maximum value of \( \ell \) is important. If we choose \( \ell \) too large, the rounding errors are bigger. If we choose it too small, we don’t have enough terms to give an accurate cross section.
2) A better way is to use the full spherical Bessel function for matching. The formula for the spherical Bessel function is given above in terms of the regular Bessel functions (besselj). The spherical Neumann functions are related to the Bessel functions of the second kind (bessely) by:

\[ r n_\ell (k r) = \frac{\pi}{2k} Y_{\ell+1/2}(k r) \]

At the matching radius \( R_M \) we can let

\[ \frac{\varphi_\ell (k R_M)}{d r \varphi_\ell (k R_M)} = \frac{g_\ell (k R_M)}{d r g_\ell (k R_M)} \]

where

\[ g_\ell (k r) = \cos \delta_\ell [r j_\ell (k r)] + \sin \delta_\ell [r n_\ell (k r)]. \]

The derivatives of \( g_\ell (k R_M) \) are best taken numerically.

A MATLAB note: To find the values of \( P_\ell (k r) \), you can use the function `legendre`, but this gives values of all the associated Legendre functions as well. You can do the following:

```matlab
theta=linspace(0,2*pi,180);
l=2;
x=cos(theta);
a=legendre(l,x);
Pl=a(1,:);
```

Given 2/10

17. Using the same nucleus and the same potential as in Problem 16, calculate the differential cross section in the First Born approximation.

18. Using the WKB relations for connecting wave functions at classical turning points, show that

\[
\begin{pmatrix}
A \\
B
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
2\Phi + \frac{1}{2\Phi} & i \left( 2\Phi - \frac{1}{2\Phi} \right) \\
-i \left( 2\Phi - \frac{1}{2\Phi} \right) & 2\Phi + \frac{1}{2\Phi}
\end{pmatrix} \begin{pmatrix}
F \\
G
\end{pmatrix}
\]

where \( \Phi \equiv \exp(\int_a^b \kappa(x) dx) \).

19. (a) Using Figure 18.2, estimate the depth of potential wells for \( \alpha \) particles in \(^{210}\text{Po}\) and \(^{238}\text{U}\). (b) Using the formula \( R=1.4 \ A^{1/3} \), estimate the radius, \( a \), of the potential wells for these two nuclei. (c) Consider an \( \alpha \) particle trapped in a potential well given by the square well described by (a) and (b) for \( r < a \) and a Coulomb potential for \( r > a \). (Remember the charge of an \( \alpha \) particle
is 2e.) You may omit the centrifugal potential from this calculation. Plot the total potential energy \( V(r) \) in MeV with \( r \) varying from 0 to 100 fm. (d) Guess a trial energy, \( E \), for each \( \alpha \) particle. Find the second classical turning point, \( b \), for this energy. Graphically estimate the second turning point, \( b \). (e) Consider the radial solution of the three-dimensional Schrödinger equation to be:

\[
\psi_3 = A(r) \frac{e^{+i kr}}{kr} + B(r) \frac{e^{-i kr}}{kr} \quad r > b \\
\psi_2 = C(r) \frac{e^{+i kr}}{kr} + D(r) \frac{e^{-i kr}}{kr} \quad b > r > a \\
\psi_1 = F(r) \frac{e^{+i kr}}{kr} + G(r) \frac{e^{-i kr}}{kr} \quad b > r > a
\]

Find and plot \( k(r) \) and \( \kappa(r) \) (on one plot) in SI units for \( r = 0 \) to 150 fm. (f) Construct a piecewise constant potential with a step size of 0.01 fm. Start with a large value of \( r \) and let \( A(r) = 1 \) and \( B(r) = 0 \). Construct a radial wave function by making the function and first derivatives continuous on the boundary. Continue back to \( r = 0 \). (g) Plot the wave functions. How were your guesses for the energies? (h) Estimate the value of \( T \), the transmission probability, for the nucleus with your best guess for the energy. Note that for the radial wave functions:

\[
T = \frac{|r \psi_{\text{trans}}|^2 k_{\text{trans}}}{|r \psi_{\text{inc}}|^2 k_{\text{inc}}} = \frac{|A|^2 k_1}{|F|^2 k_\infty}
\]

Notes:
1) I found it easier to put everything in SI units.
2) As you approach the classical turning points, \( kr \to 0 \), and the equations you use to connect the coefficients become invalid as they have \( k \) in the denominator. At \( r = a \), this isn’t much of a problem as the potential is discontinuous there and \( kr \) never gets small. At \( r = b \), it is best to take a point or two to either side of \( b \), let \( k = 0 \) in that region, and use \( \psi = A + \frac{B}{r} \) there. (This is the solution to Laplace’s equation in spherical coordinates for \( \ell = 0 \).)