Physics 416 Assignment L2

“Does the Inertia of a Body Depend on Its Energy Content?”

This is the paper in which Einstein first introduced the concept of $E = mc^2$, though not in those terms. In addition to being of historical interest, it provides a nice example of a succinct physics paper.

1. Read the entire paper. See how much you understand on a first pass.
2. What did Einstein assume you already knew before you started reading the paper?
   Did you already know it?

Einstein started with electromagnetic theory to get the first equation in his paper, the one for the transformation of the energy of light. Let’s write this equation in more modern notation. Let $E$ be the energy of a photon in a reference frame $S$. A second frame $S'$ moves at a speed $v$ with respect to $S$ in the $+x$-direction. If the photon moves at an angle $\phi$ with respect to the $x$-axis in the $S$ frame, the energy of the photon in $S'$ is:

$$E' = \gamma E (1 - \beta \cos \phi)$$

where

$$\beta = \frac{v}{c}$$
$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

Now, let’s use Einstein’s notation:

$$E' \rightarrow l^*$$
$$E \rightarrow l$$
$$c \rightarrow V$$
$$\beta \rightarrow \frac{v}{V}$$
$$\gamma \rightarrow \frac{1}{\sqrt{1 - (\frac{v}{V})^2}}.$$

The energy relation for light becomes:

$$l^* = l \frac{1 - \frac{v}{V} \cos \phi}{\sqrt{1 - \left(\frac{v}{V}\right)^2}}.$$ 

Now all we need to remember is that the total energy is the sum of the kinetic energy and the rest energy for a body.
We now return to the assignment:

3. Reread the paper, paying careful attention to the logic in the section after Einstein introduces the energy equation for light. Write a flowchart of this logic. You don’t need to submit the chart, but we will talk about it in class.

It may be helpful to consider what this paper means from a present-day viewpoint.

Let us assume that an event (think of a little explosion) occurs in reference frame $S$ at location $(x,y,z)$ and time $ct$. Note that we have multiplied the time by the speed of light so that it has dimensions of length. A second frame $S'$ moves at a speed $v$ in the $+x$-direction. The Lorentz transformation equations are:

\[\begin{align*}
ct' &= \gamma ct - \beta \gamma x \\
x' &= \gamma x - \beta \gamma ct \\
y' &= y \\
z' &= z
\end{align*}\]

A consequence of Einstein’s argument in this paper is that the total energy $E$ and the momentum $pc$ of objects transform in a similar fashion:

\[\begin{align*}
E' &= \gamma E - \beta \gamma p_x c \\
p_x' &= \gamma p_x - \beta \gamma E \\
p_y' &= p_y \\
p_z' &= p_z
\end{align*}\]

If $\varphi$ is the angle between the $x$-axis and the direction of the momentum, we can write $p_x$ as $p \cos \varphi$.

If we start from this answer, we can go backward and find the transformation equations for photons. For a photon, $E = \sqrt{p^2 c^2 + m_0^2 c^4} = pc$. The first equation of the energy-momentum transformations reduces to

\[E' = \gamma E - \beta \gamma E \cos \varphi,\]

the equation with which Einstein began.
the additional notes to this paper, the contents of some of the notes suggest that he was consulted.

[2] Einstein introduces the designations $\Xi$, $H$, $Z$ for the coordinates for the $x'$, $y'$, $z'$ axes of the moving system.

[3] In the 1913 reprint, the following note is appended to the end of this line: “The Lorentz transformation equations are more simply derivable directly from the condition that, as a consequence of these equations, the relation $\xi^2 + \eta^2 + \zeta^2 = V^2 \gamma^2 = 0$ shall have the other $x^2 + y^2 + z^2 - V^2 t^2 = 0$ as a consequence.”


[5] This result later became known as “the clock paradox.” In 1911, Langevin seems to have first introduced human travelers, leading to the alternate name, “the twin paradox.”

[6] In the 1913 reprint, the following note is appended to the word “Unruhuhub”: “In contrast to the ‘pendulum clock,’ which—from the physical standpoint—is a system, to which the earth belongs; this had to be excluded.”

[7] This fraction should be $\frac{a}{\sqrt{c^2}}$.

[8] The term “motional magnetic force” was introduced by Heaviside. Einstein later defined the “magnetomotive force” as the force acting on a unit of magnetic charge moving through an electric field. To the order of approximation used in the discussion of “electromotive force,” the magnetomotive force is given by $-1 / V \{v, E\}$, where $E = (L, M, N)$, $v = (u, 0, 0)$, and the bracket is a vector product.

[9] Corrected by Einstein in a reprint copy to “for $v = -V, \nu = -\infty$.”

[10] In ibid., “the connecting line ‘light source—observer’” was canceled and interlineated with “direction of motion.”

[11] $\alpha$ should be $\varphi$.

[12] In a reprint copy, the denominator in the final term is corrected to “$1 - (v/c)^2$.”

[13] In the 1913 reprint, the following note is appended to “call”’: “The definition of force given here is not advantageous as was first noted by M. Planck. It is instead appropriate to define force in such a way that the laws of momentum and energy conservation take the simplest form.”

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**PAPER 4**

**Does the Inertia of a Body Depend on Its Energy Content?**

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The results of an electrodynamic investigation recently published by me in this journal lead to a very interesting conclusion, which will be derived here.

I based this investigation on the Maxwell-Hertz equations for empty space, together with Maxwell's expression for the electromagnetic energy of space, and also the following principle:

The laws according to which the states of physical systems change are independent of which one of the two coordinate systems (assumed to be in uniform parallel-translational motion relative to each other) is used to describe these changes (the principle of relativity).

Based on this foundation, I derived the following result, among others (loc. cit., sec. 8).

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2 The principle of the constancy of the velocity of light used there is of course contained in Maxwell’s equations.
Let a system of plane light waves have the energy \( l \) relative to the coordinate system \((x, y, z)\); let the ray direction (the wave-normal) make the angle \( \varphi \) with the \( x \)-axis of the system. If we introduce a new coordinate system \((\xi, \eta, \zeta)\), which is in uniform parallel translation with respect to the system \((x, y, z)\), and the origin of which moves along the \( x \)-axis with velocity \( v \), then this quantity of light—measured in the system \((\xi, \eta, \zeta)\)—has the energy

\[
I^* = I \frac{1 - \frac{v}{V} \cos \varphi}{\sqrt{1 - \left(\frac{v}{V}\right)^2}},
\]

where \( V \) denotes the velocity of light. We shall make use of this result in what follows.

Let there be a body at rest in the system \((x, y, z)\) whose energy, relative to the system \((x, y, z)\), is \( E_0 \). Let the energy of the body be \( H_0 \), relative to the system \((\xi, \eta, \zeta)\), moving with velocity \( v \) as above.

Let this body emit plane light waves of energy \( L/2 \) (measured relative to \((x, y, z)\)) in a direction forming an angle \( \varphi \) with the \( x \)-axis, and at the same time an equal amount of light in the opposite direction. The body remains at rest with respect to system \((x, y, z)\) during this process. This process must satisfy the principle of conservation of energy, and must be true (according to the principle of relativity) with respect to both coordinate systems. If \( E_1 \) and \( H_1 \) denote the energy of the body after emission of the light, measured relative to the system \((x, y, z)\) and the \((\xi, \eta, \zeta)\), respectively, we obtain, using the relation indicated above,

\[
E_0 = E_1 + \left[ \frac{L}{2} + \frac{L}{2} \right],
\]

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\[
H_0 = H_1 + \left[ \frac{L}{2} \frac{1 - \frac{v}{V} \cos \varphi}{\sqrt{1 - \left(\frac{v}{V}\right)^2}} \right] + \left[ \frac{L}{2} \frac{1 + \frac{v}{V} \cos \varphi}{\sqrt{1 - \left(\frac{v}{V}\right)^2}} \right]
\]

\[
= H_1 + \frac{L}{\sqrt{1 - \left(\frac{v}{V}\right)^2}}.
\]

By subtraction, we obtain from these equations

\[
(H_0 - E_0) - (H_1 - E_1) = L \left( \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}} - 1 \right).
\]

Both differences of the form \( H - E \) occurring in this expression have simple physical meanings. \( H \) and \( E \) are the energy values of the same body, related to two coordinate systems in relative motion, the body being at rest in one of the systems (system \((x, y, z)\)). Hence it is clear that the difference \( H - E \) can differ from the body's kinetic energy \( K \) with respect to the other system (system \((\xi, \eta, \zeta)\)) only by an additive constant \( C \), which depends on the choice of the arbitrary additive constants in the energies \( H \) and \( E \). We can therefore set

\[
H_0 - E_0 = K_0 + C
\]

\[
H_1 - E_1 = K_1 + C,
\]

since \( C \) does not change during the emission of light. So we get

\[
K_0 - K_1 = L \left( \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}} - 1 \right).
\]

The kinetic energy of the body with respect to \((\xi, \eta, \zeta)\) decreases as a result of emission of the light by an amount
INERTIA OF A BODY ENERGY CONTENT

\[ H_0 = H_1 + \left( L \frac{1 - \frac{V}{V} \cos \varphi}{\sqrt{1 - \left( \frac{V}{V} \right)^2}} + L \frac{1 + \frac{V}{V} \cos \varphi}{\sqrt{1 - \left( \frac{V}{V} \right)^2}} \right) \]

\[ = H_1 + \frac{L}{\sqrt{1 - \left( \frac{V}{V} \right)^2}}. \]

By subtraction, we obtain from these equations

\[ (H_0 - E_0) - (H_1 - E_1) = L \left\{ \frac{1}{\sqrt{1 - \left( \frac{V}{V} \right)^2}} - 1 \right\}. \]

Both differences of the form \( H - E \) occurring in this expression have simple physical meanings. \( H \) and \( E \) are the energy values of the same body, related to two coordinate systems in relative motion, the body being at rest in one of the systems (system \((x, y, z)\)). Hence it is clear that the difference \( H - E \) can differ from the body's kinetic energy \( K \) with respect to the other system (system \((\xi, \eta, \zeta)\)) only by an additive constant \( C \), which depends on the choice of the arbitrary additive constants in the energies \( H \) and \( E \). We can therefore set

\[ H_0 - E_0 = K_0 + C, \]
\[ H_1 - E_1 = K_1 + C, \]

since \( C \) does not change during the emission of light. So we get

\[ K_0 - K_1 = L \left\{ \frac{1}{\sqrt{1 - \left( \frac{V}{V} \right)^2}} - 1 \right\}. \]

The kinetic energy of the body with respect to \((\xi, \eta, \zeta)\) decreases as a result of emission of the light by an amount...
that is independent of the properties of the body. Furthermore, the difference \( K_0 - K_1 \) depends on the velocity in the same way as does the kinetic energy of an electron (loc. cit., sec. 10).

Neglecting magnitudes of the fourth and higher order, we can get\(^1\)

\[
K_0 - K_1 = \frac{L}{V^2} \frac{v^2}{2}.
\]

From this equation one immediately concludes:

If a body emits the energy \( L \) in the form of radiation, its mass decreases by \( L/V^2 \). Here it is obviously inessential that the energy taken from the body turns into radiant energy, so we are led to the more general conclusion:

The mass of a body is a measure of its energy content; if the energy changes by \( L \), the mass changes in the same sense by \( L/9 \cdot 10^{20} \) if the energy is measured in ergs and the mass in grams.

It is not excluded that it will prove possible to test this theory using bodies whose energy content is variable to a high degree (e.g., radium salts).

If the theory agrees with the facts, then radiation carries inertia between emitting and absorbing bodies.

(Annalen der Physik 18 [1905]: 639–641)

EDITORIAL NOTES

\(^1\)Einstein used the Newtonian limit of the body's kinetic energy in order to evaluate change in its rest mass.

Part Four

Einstein's Early Work on the Quantum Hypothesis