Physics 360
Review 4

The test will be similar to the previous tests. Calculators will not be allowed, the Unit #3 material will be divided into three different parts, there will be one problem from the Unit #1 test, one from the Unit #2 test, and one from the Unit#3 test.

Information that will be provided

$F = U - TS, \quad H = U + PV, \quad G = U + PV - TS$

$$\left(P + \frac{aN^2}{V^2}\right)(V - Nb) = NkT$$

$$e = 1 - \frac{T_C}{T_H}$$

At constant energy and volume, S tends to increase.
At constant temperature and volume, F tends to decrease.
At constant temperature and pressure, G tends to decrease.

$$\mu(P,T) = \mu^o(T) + kT\ln\left(\frac{P}{P^o}\right)$$

$$\mu_A(P,T) = \mu^o(P,T) - \frac{N_B kT}{N_A}$$

$$\mu_B(P,T) = f(P,T) + kT\ln m_B$$

Some of the following may be given, but you may be asked to prove them:

$dU = T\,dS - P\,dV + \mu\,dN$
$dH = T\,dS + V\,dP + \mu dN$
$dF = -P\,dV + \mu\,dN - S\,dT$
$dG = -S\,dT + V\,dP + \mu\,dN$

$S = -\left(\frac{\partial G}{\partial T}\right)_{P,N}$, etc. There are 24 such equations that can be easily derived from the thermodynamic identities.

Three integrals involving Gaussian functions:

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad \int_0^\infty xe^{-ax^2}dx = \frac{1}{2a}, \quad \int_0^\infty x^2e^{-ax^2}dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \quad a > 0$$

Summing a geometric series:

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-x}$$

Section I. Review Questions
(3 problems, 12 points each)

Taken from the first three exams. See solutions online.

Section II. Qualitative Questions
(4 problems, 8 points each)

Understand the following basic ideas. Be sure to be able to describe them in terms of concepts as well as in terms of equations.

Boltzmann factor and the partition function: how they relate to probability in a system where the number of particles is fixed.
Gibbs factor and the grand partition function: how they relate to probability in a system where the number of particles is not fixed.
Know the definition of mean values and standard deviations. Be able to describe them in words.
Bosons (integral spin) and fermions (half-integral spin) and how their statistics differ from Maxwell-Boltzmann statistics.
Fermi distribution function: functional form, meaning of "skin thickness."
BE and FD distributions, average occupation numbers. Know the following equations:

$$\bar{n}_{FD} = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}, \quad \bar{n}_{BE} = \frac{1}{e^{(\varepsilon-\mu)/kT}-1}, \quad \bar{n}_{MB} = \frac{1}{e^{(\varepsilon-\mu)/kT}},$$

Degenerate fermi gas (near zero degrees).
Density of states $g(\varepsilon)$. The number of states between ε and ε $+d\varepsilon$.
Know that the chemical potentials for phonons and photons are zero.
Blackbody: completely absorbs incident radiation.
Stefan-Boltzmann law: the total energy radiated is proportional to T^4.
Phonons: lattice vibrations with quantized energies.
Debye temperature: roughly, above the Debye temperature, one can use the equipartition theorem for C_V.
Bose-Einstein condensation: below T_c, the ground state starts becoming populated with a significant fraction of the total atoms. Above T_c, the ground state is the most highly populate, but the fraction of total atoms occupying it is still very small.

Section III. Basic Problems
(4 problems, 8 points each)

Be able to find the (grand) partition function for a system if I give you the states and their energies. (6.1 and 7.1)
Know the states and energies for two-state paramagnets so you can find the partition function yourself in this case.
Use the partition function to calculate average energies, standard deviations of energies, and probabilities of finding a system in given states. Know the relationship $\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2$. (6.2)
Be able to turn the partition function to an integral where warranted as the book does with quantum mechanical rotators (6.2) and the ideal gas (6.7)
Be able to prove that the equipartition function holds for systems with quadratic degrees of freedom. (6.3)

Be able to derive the Maxwell speed distribution $D(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-mv^2/2kT}$ by constructing a partition function and normalizing the distribution to unity.

Given the Maxwell speed distribution, be able to use it to calculate mean velocities and standard deviations.
Given the Maxwell speed distribution, write down the distribution for v_x.
Be able to derive Eq. (6.79) and to give written justifications for each step.
Use the ideal gas partition function and thermodynamic identities to derive equations for pressure, C_V, etc. (6.7)
Given a partition function, find the Helmholtz free energy and use thermodynamic identities to determine S, P, and μ. (6.5)
Be able to write down available states for simple systems of bosons, fermions, and distinguishable particles.
Derive the BE and FD distribution functions by using a grand partition function. (7.2)
Be able to derive Eq. (7.42) for a zero-degree fermi gas.
Be able to derive Eq. (7.51) for the density of states.
(You need not review the Sommerfeld expansion in 7.3.)
Be able to reproduce the argument that leads to Eq. (7.80), and apply this to derive the energy density of the electromagnetic spectrum, Eq. (7.84).
(You need not review photons escaping from a hole nor solar radiation from 7.4.)
(Chapter 8 will not be covered.)

Section IV. Synthesis Problems
(2 problems, 10 points each)

Sample Test

Section I. Review Question 1–3
See the previous tests.

Section II. Qualitative Questions

4. What is the difference between a partition function and a grand partition function?

5. Sketch a Gaussian distribution. Label the mean value and the standard deviation on your graph. Write the functional form of the Gaussian in terms of the mean and standard deviation. Be sure it is normalized to unit probability.

6. What is meant by the skin thickness of fermi distribution? Write down the mathematical form of the function and identify the skin thickness in your equation.

7. Describe what is meant by Bose-Einstein condensation.

Section III. Basic Problems

8. The energy of a system is given by the expression $E(q) = cq^2$ where q is a discrete variable that can take on any value from 0 to infinity.
(a) Construct a partition function for the system.
(b) Find the average energy for the system.

9. The Maxwell speed distribution in three dimensions is given by the equation

$$D(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-mv^2/2kT}$$

Find the average value of $|v_x|$?

10. We may treat electrons in a solid as a fermi gas. Take the Fermi energy to be ln(3) in eV. At room temperature, how does the probability of finding an electron with an energy at 10% above the Fermi energy compare with the probability of finding it at very low energy? You may use the approximation that kT =1/40 eV at room temperature. How does your answer change if you take electron spin into consideration?

11. Show that the energy density of photons emitted from a black body is

$$u(\varepsilon) = \frac{8\pi}{(hc)^3}\frac{\varepsilon^3}{e^{\varepsilon/kT} - 1}$$

Section IV. Synthesis Problems 12–13

12. Consider a paramagnet with spin 3/2 in an external magnetic field B. A single paramagnet then can be in any of four states with spins and energies given in the table below.

Spin	Energy
–3/2	$+3/2\ \mu B$
–1/2	$+1/2\ \mu B$
1/2	$-1/2\ \mu B$
3/2	$-3/2\ \mu B$

(a) Find the partition function for a single paramagnet.
(b) What is the ratio of the probability of finding the paramagnet with spin +3/2 to the probability of finding it with spin –3/2?
(c) Find the average energy of the paramagnet.
(d) The magnetization of a system of N such paramagnets is given by the total energy multiplied by $-1/B$. Show that the magnetization is proportional to $1/T$ when T is large.

13. A system of N particles each has a single-particle partition function, Z_1.

(a) Using Boltzmann statistics, find an expression for the chemical potential μ in terms of Z_1 by making use of the Helmholtz free energy.
Hints: Don't forget to take multiplicity into account. Use Stirling's Formula to simplify your answer.

b) Now sum over the MB distribution function to get the total number of particles. Find the condition on μ that makes this expression true. (You won't get any points for guessing, even if your guess is right, so be sure to show your work!)

Hint: Be looking for the single-particle partition function.

Selected Solutions

12. a) Find the partition function for a single paramagnet.

$$Z = e^{-3\mu B/2kT} + e^{-1\mu B/2kT} + e^{+1\mu B/2kT} + e^{+3\mu B/2kT}$$

(b) What is the ratio of the probability of finding the paramagnet with spin +3/2 to the probability of finding it with spin –3/2?

$$\frac{P(+\frac{3}{2})}{P(-\frac{3}{2})} = \frac{e^{+3\mu B/2kT}}{e^{-3\mu B/2kT}} = e^{+6\mu B/2kT}$$

(c) Find the average energy of the paramagnet.

$$\langle E \rangle = \frac{\frac{3}{2}\mu B e^{-3\mu B/2kT} + \frac{1}{2}\mu B e^{-1\mu B/2kT} - \frac{1}{2}\mu B e^{+1\mu B/2kT} - \frac{3}{2}\mu B e^{+3\mu B/2kT}}{e^{-3\mu B/2kT} + e^{-1\mu B/2kT} + e^{+1\mu B/2kT} + e^{+3\mu B/2kT}}$$

(d) The magnetization of a system of N such paramagnets is given by the total energy multiplied by $-1/B$. Show that the magnetization is proportional to $1/T$ when T is large.

$$M = -\frac{N}{B}\frac{\mu B}{2}\frac{3e^{-3\mu B/2kT} + e^{-1\mu B/2kT} - e^{+1\mu B/2kT} - 3e^{+3\mu B/2kT}}{e^{-3\mu B/2kT} + e^{-1\mu B/2kT} + e^{+1\mu B/2kT} + e^{+3\mu B/2kT}}$$

$$\approx -\frac{N\mu}{2}\frac{3(1-3\mu B/2kT) + (1-\mu B/2kT) - (1+\mu B/2kT) - 3(1+3\mu B/2kT)}{(1-3\mu B/2kT) + (1-\mu B/2kT) + (1+\mu B/2kT) + (1+3\mu B/2kT)}$$

$$= -\frac{N\mu}{2} \times \frac{-9\mu B/2kT - \mu B/2kT - \mu B/2kT - 9\mu B/2kT}{4}$$

$$= \frac{5N\mu^2 B}{4kT}$$

13.(a) Using Boltzmann statistics, find an expression for the chemical potential μ in terms of Z_1, T, and N by making use of the Helmholtz free energy.
Hints: Don't forget to take multiplicity into account. Use Stirling's Formula to simplify your answer.

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -kT\left(\frac{\partial \ln Z}{\partial N}\right)_{T,V}$$

$$= -kT\left(\frac{\partial \ln[Z_1^N / N!]}{\partial N}\right)_{T,V}$$

$$= -kT\left(\frac{\ln Z_1 \partial N}{\partial N} - \frac{\partial}{\partial N}[N \ln N - N]\right)_{T,V}$$

$$= -kT \ln\frac{Z_1}{N}$$

(b) Now sum over the MB distribution function to get the total number of particles. Find the condition on μ that makes this expression true.

Hint: Be looking for the single-particle partition function!

$$N = \sum_s \frac{1}{e^{(\varepsilon_s - \mu)/kT}} = e^{\mu/kT} \sum_s e^{-\varepsilon_s/kT} = e^{\mu/kT} Z_1$$

$$e^{\mu/kT} = \frac{N}{Z_1}$$

$$\mu = kT \ln \frac{N}{Z_1} = -kT \ln \frac{Z_1}{N}$$