The Hamiltonian

- \( H = H(x, p, t) \) whereas \( L = L(x, \dot{x}, t) \)
- For simple systems, \( H = T + U \)
- There are two \textit{first order} equations of motion for each variable:
  \[
  \frac{\partial H}{\partial x} = -\dot{p} \quad \frac{\partial L}{\partial \dot{x}} = \dot{p} \\
  \frac{\partial H}{\partial p} = +\dot{x} \quad \left( \frac{\partial L}{\partial \dot{x}} = p \right)
  \]
- The Lagrangian method gives one \textit{second order} equation for each variable.

The Hamiltonian - Origins

Take \( L = L(q_1, q_2, q_3, t) \)

\[
\begin{align*}
\frac{dL}{dt} & = \frac{\partial L}{\partial q_1} \dot{q}_1 + \frac{\partial L}{\partial q_2} \dot{q}_2 + \frac{\partial L}{\partial q_3} \dot{q}_3 + \frac{\partial L}{\partial t} \\
& = p_1 \dot{q}_1 + p_2 \dot{q}_2 + \frac{\partial L}{\partial t} \\
& = \frac{d}{dt} (p_1 \dot{q}_1 + p_2 \dot{q}_2) + \frac{\partial L}{\partial t}
\end{align*}
\]

Therefore

\[
\frac{d}{dt} (p_1 \dot{q}_1 + p_2 \dot{q}_2 - L) + \frac{\partial L}{\partial t} = 0
\]

The Hamiltonian - Notes

- The Hamiltonian is a function of \( p \) and \( q \). But \( p \) is not ‘the momentum,’ it is the generalized momentum conjugate to \( q \).
  \[ p_i = \frac{\partial L}{\partial \dot{q}_i} \]
- The general expression is:
  \[ H = \sum_i p_i \dot{q}_i - L \]
- That means you generally have to find the Lagrangian before you can find he Hamiltonian!

The Hamiltonian - Notes

- The Hamiltonian is \( H=T+U \) unless
  - The Lagrangian has explicit time dependence
  - The transformation between the coordinates \( q_i \) and Cartesian coordinates have explicit time dependence
- But ... you have to write the Hamiltonian in terms of the correct generalized momenta – so you usually still need the Lagrangian first!
Example

hamilton.nb