Physics 321

Hour 16
Lagrangian Dynamics

Bottom Line

- The Lagrangian is defined as $\mathcal{L} = T - U$
- When we minimize the “action integral”
  $$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt,$$
  the value of $x(t)$ that is given is the actual motion of an object.
- This means that
  $$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$
- This is all magic.

Changing the Names...

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt$$

$S$ is an extremum (so varying the path a little does not change $S$) if

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

Is there a function $\mathcal{L}$ such that $S$ is stationary for the actual $x(t)$ chosen by nature?

-- Comparison with results from “standard” Newtonian mechanics shows that there is!

Changing the Names...

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt$$

The Lagrangian

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt$$

$S$ is called the “action integral” and $\mathcal{L}$ is the “Lagrangian”

$$\mathcal{L} = T - U$$

Using The Lagrangian

- Find the kinetic and potential energies
- Construct the Lagrangian
  $$\mathcal{L} = T - U$$
- Use Lagrange’s Equation
  $$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$
- We never need to consider the action integral
Using The Lagrangian

- Variable = \( q \) (like \( x \) or \( \theta \))
- Generalized momentum (like \( p \) or \( L \))

\[ p = \frac{\partial L}{\partial \dot{x}} \]

- Lagrange's Equation

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \]

\[ \dot{p} = \frac{\partial L}{\partial q} \]

Examples
- A mass \( m \) falls freely
- A mass \( m \) is acted on by a spring with constant \( k \) (no gravity)
- Add gravity
- Add friction

Using The Lagrangian

\[ \dot{p} = \frac{\partial L}{\partial q} \]

In a simple case \( \dot{p} \) is the force as is \( \frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q} \)

The Lagrangian and Mathematica

pendulum.nb
spring pendulum.nb
parabolic bowl.nb