Physics 321

Hour 15
The Calculus of Variations

Bottom Line

- A simple derivative is useful for finding the value of a variable that maximizes or minimizes a function.
- The calculus of variations is useful for finding the function that maximizes or minimizes a quantity that depends on the function.
- You don’t need to reproduce this section, but it is an important tool in physics that you should put forth some real effort to understand.

The Life Guard Problem

You are a life guard on a beach and see someone needing your help. You want to get there as fast as possible, where do you jump in the water?

Example

SnellsLaw.nb

Finding the Shortest Path between Points

The path length is

\[ L = \int_{P_1}^{P_2} ds \]

\[ ds = \sqrt{dx^2 + dy^2} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx \sqrt{1 + y'^2} \]

\[ L = \int_{P_1}^{P_2} \sqrt{1 + y'^2} \, dx \]

Finding the Shortest Path between Points

The path length is

\[ L = \int_{P_1}^{P_2} \sqrt{1 + y'^2} \, dx \]

Take another function \( Y(x) = y(x) + a\eta(x) \) subject to \( \eta(x_1) = \eta(x_2) = 0 \).

What does that tell us about \( Y(x) \)?
Finding the Shortest Path between Points

If we choose an arbitrary path \( Y(x) = y(x) + a\eta(x) \), where \( \eta(x_1) = \eta(x_2) = 0 \), then \( y(x) \) is the shortest path if

\[
\frac{\partial L}{\partial \alpha} \bigg|_{\alpha=0} = 0
\]

for all functions \( \eta(x) \).

A Useful Theorem

Integration by parts give us:

\[
\int_A^B u dv = uv \bigg|_A^B - \int_A^B v du
\]

Let \( du = \frac{du}{dx} dx = u' dx \) \( dv = v' dx \)

\[
\int_A^B uv' dx = uv \bigg|_A^B - \int_A^B v u' dx
\]

Distance Between Two Points

\[
L = \int_{P_1}^{P_2} ds
\]

\[
ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'^2}
\]

Let \( Y(x) = y(x) + a\eta(x) \) where \( \eta(x_1) = \eta(x_2) = 0 \)

\[
L(\alpha) = \int_{x_1}^{x_2} \sqrt{1 + Y'(x)^2} \ dx
\]

Note \( \frac{\partial f}{\partial \alpha} = \eta' \frac{\partial f}{\partial y'} \)
Distance Between Two Points

\[
\frac{\partial L}{\partial \alpha} = \int_{x_1}^{x_2} \eta \frac{\partial f}{\partial y'} \, dx
\]

\[
= \left[ \eta \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx} \frac{\partial f}{\partial y'} \, dx = 0
\]

\[
\frac{d}{dx} \frac{\partial f}{\partial y'} = 0
\]

Distance Between Two Points

\[
\frac{\partial f}{\partial y'} \bigg|_{\alpha \to 0} = \frac{2(y' + \alpha \eta')}{2\sqrt{1 + (y' + \alpha \eta')^2}} \bigg|_{\alpha \to 0}
\]

\[
= \frac{y'}{\sqrt{1 + (y')^2}} = \text{constant}
\]

\[
\rightarrow y'(x) = \text{constant}
\]

\[
\rightarrow y(x) = mx + b
\]

The Action Integral

\[
S = \int_{t_1}^{t_2} L(x, \dot{x}, t) \, dt
\]

\[
S(\alpha) = \int_{t_1}^{t_2} L(x + \alpha \eta, \dot{x} + \alpha \dot{\eta}, t) \, dt
\]

\[
\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial r} \frac{\partial r}{\partial \alpha} + \frac{\partial L}{\partial s} \frac{\partial s}{\partial \alpha} \right] dt = 0
\]

\[
\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial x} \eta - \frac{\partial L}{\partial \dot{x}} \dot{\eta} \right] dt = 0
\]

The Action Integral

\[
\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial x} \eta - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right] dt = 0
\]

\[
\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right] dt = 0
\]

\[
\rightarrow \frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}
\]