

Dealing with Rotating Coordinate Systems

Physics 321

The treatment of rotating coordinate frames can be very confusing because there are two different sets of axes, and one set of axes is not constant in time. This handout is an attempt to clarify the problems of transforming position, velocity, and acceleration in a fixed frame of reference to a rotating frame of reference.

Because we are dealing with rotations, it is useful to write the vectors as column vectors so that we can apply rotation matrices to them. Usually we just write:

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\hat{x} + y\hat{y} + z\hat{z},$$

but because the basis vectors are implied and not explicitly written, we are going to write the same thing as a product (the dot product) of a basis set and the components in that basis. Thus:

$$\vec{r} = (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\hat{x} + y\hat{y} + z\hat{z},$$

Let's take two sets of coordinate axes that have the same origin. There exists a linear transformation that relates one set of basis vectors to the other set of basis vectors. Since the lengths of the basis vectors (they're all unit vectors) do not change, we can think of this transformation as a rotation, or actually three rotations around different axes in general). This we can write:

$$\begin{aligned} \text{(Eq. 1)} \quad (\hat{x}' \quad \hat{y}' \quad \hat{z}') &= (\hat{x} \quad \hat{y} \quad \hat{z})\mathbf{R} \quad \text{or} \\ (\hat{x} \quad \hat{y} \quad \hat{z}) &= (\hat{x}' \quad \hat{y}' \quad \hat{z}')\mathbf{R}^{-1} \end{aligned}$$

where the primes indicate quantities in the rotating coordinate system and the unprimed quantities represent quantities in the fixed coordinate frame. Note that the rotation matrix follows the vector because we are rotating a row vector. We'll later see how to write \mathbf{R} in terms of rotation angles.

A position vector is just an object defined by two points in space. It can be described without a coordinate system of any kind, or it can be written in terms of components. Figure 1 illustrates this idea.

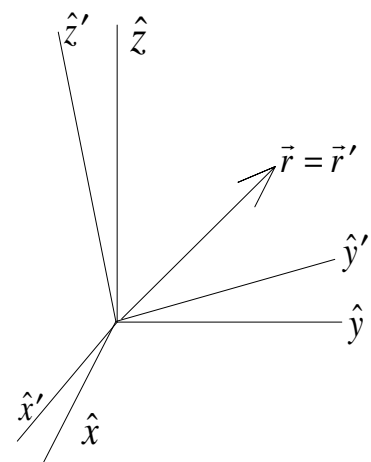


Figure 1 A vector in terms of two different sets of basis vectors.

Since the vector is the same object in either coordinate system, we can write:

$$\vec{r} = \vec{r}'$$

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Now we can do some manipulating with the rotation matrix. Noting that any matrix times its inverse is the identity:

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \mathbf{R}^{-1} \mathbf{R} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Because of Eq.1, we may write:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{R} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{R}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So now we can transform a position vector – both components and unit vectors – from the fixed to the rotating frame, as long as we know the rotation matrix that relates the two frames to each other.

Now let's look at velocities:

$$\frac{d}{dt} \vec{r} = \frac{d}{dt} \vec{r}'$$

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{\hat{x}}' & \dot{\hat{y}}' & \dot{\hat{z}}' \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix}$$

The only hard part here is taking the derivatives of the primed unit vectors.

$$\frac{d}{dt} \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \mathbf{R} = \begin{pmatrix} \dot{\hat{x}} & \dot{\hat{y}} & \dot{\hat{z}} \end{pmatrix} \mathbf{R}$$

$$\begin{pmatrix} \dot{\hat{x}}' & \dot{\hat{y}}' & \dot{\hat{z}}' \end{pmatrix} = \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \mathbf{R}^{-1} \dot{\mathbf{R}}$$

Armed with these, we can write the equations that transform the components of velocity from one coordinate frame to the other. First, let us put everything in terms of the unprimed basis vectors.

$$\begin{aligned}
 (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= (\hat{x} \quad \hat{y} \quad \hat{z}) \dot{\mathbf{R}} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + (\hat{x} \quad \hat{y} \quad \hat{z}) \mathbf{R} \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix} \\
 (\text{Eq. 4}) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= \dot{\mathbf{R}} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \mathbf{R} \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix}
 \end{aligned}$$

This expression then gives the components of velocity in the fixed system in terms of the rotation matrix and the components of position and velocity in the unprimed system.

We can similarly find the components of the velocity in the primed system in terms of the components of position and velocity in the unprimed system.

$$\begin{aligned}
 (\hat{x}' \quad \hat{y}' \quad \hat{z}') \mathbf{R}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= (\hat{x}' \quad \hat{y}' \quad \hat{z}') \mathbf{R}^{-1} \dot{\mathbf{R}} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + (\hat{x}' \quad \hat{y}' \quad \hat{z}') \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix} \\
 (\text{Eq. 5}) \quad \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix} &= \mathbf{R}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} - \mathbf{R}^{-1} \dot{\mathbf{R}} \mathbf{R}^{-1} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}
 \end{aligned}$$

A more familiar way of writing these equations can be obtained by going back to the line before Eq. 3 and writing the result in mixed unit vectors:

$$\begin{aligned}
 (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= (\hat{x}' \quad \hat{y}' \quad \hat{z}') \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix} \\
 (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= (\hat{x}' \quad \hat{y}' \quad \hat{z}') \mathbf{R}^{-1} \dot{\mathbf{R}} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix} \\
 (\hat{x} \quad \hat{y} \quad \hat{z}) \vec{v} &= (\hat{x}' \quad \hat{y}' \quad \hat{z}') [\vec{v}' + \mathbf{R}^{-1} \dot{\mathbf{R}} \vec{r}'] \\
 (\hat{x} \quad \hat{y} \quad \hat{z}) \vec{v} &= (\hat{x}' \quad \hat{y}' \quad \hat{z}') [\vec{v}' + \vec{\omega}' \times \vec{r}']
 \end{aligned}$$

Here, $\vec{\omega}'$ is the angular velocity of the rotating system as measured in the fixed frame, but whose components are measured in the rotating frame, not in the fixed frame. That is:

$$\vec{\omega}' = \mathbf{R}^{-1}\vec{\omega}$$

where $\vec{\omega}$ is the angular velocity of the rotating frame as measured in the fixed frame with the unit vectors of the fixed frame.

Therefore, as long as we remember that everything on the left-hand side is expressed in terms of the unit vectors of the fixed frame, and everything on the right-hand side is expressed in terms of the rotating frame, we have:

$$(Eq. 6) \quad \vec{v} = \vec{v}' + \vec{\omega}' \times \vec{r}'$$

The term with the rotation matrices can also be written in terms of the unprimed vectors as follows:

$$\begin{aligned} \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \dot{\mathbf{R}} \mathbf{R}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \begin{pmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{pmatrix} \\ \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \vec{v}' &= \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} [\vec{v} - \dot{\mathbf{R}} \mathbf{R}^{-1} \vec{r}] \\ \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \vec{v}' &= \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} [\vec{v} - \vec{\omega} \times \vec{r}] \end{aligned}$$

$$(Eq. 7) \quad \vec{v}' = \vec{v} - \vec{\omega} \times \vec{r}$$

In these expressions, we have made use of the identities

$$\mathbf{R}^{-1} \dot{\mathbf{R}} \vec{r}' = \vec{\omega}' \times \vec{r}', \quad \dot{\mathbf{R}} \mathbf{R}^{-1} \vec{r} = \vec{\omega} \times \vec{r}.$$

Example 1:

A frictionless block moves radially outward from the center of a turntable. Let us take the velocity of the block in the lab to be $\vec{v} = v_0 \hat{x}$. Find the position and velocity of the block as measured by someone on the turntable. Take the angular speed of the turntable to be ω .

First, let's recapitulate what we know:

$$\vec{\omega} = \omega \hat{z}, \quad \vec{v} = v_0 \hat{x}, \quad \vec{r} = v_0 t \hat{x}$$

Eq. 7 gives:

$$\begin{aligned}
 (\hat{x}' \quad \hat{y}' \quad \hat{z}')\vec{v}' &= (\hat{x} \quad \hat{y} \quad \hat{z})[\vec{v} - \vec{\omega} \times \vec{r}] \\
 &= v_0\hat{x} - \omega v_0 t \hat{z} \times \hat{x} \\
 &= v_0\hat{x} - \omega v_0 t \hat{y} \\
 &= (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{bmatrix} 1 \\ -\omega t \\ 0 \end{bmatrix} v_0
 \end{aligned}$$

For position, we have:

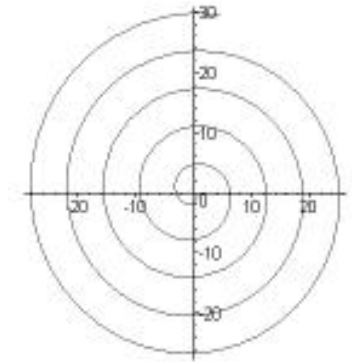
$$(\hat{x}' \quad \hat{y}' \quad \hat{z}')\vec{r}' = (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{pmatrix} v_0 t \\ 0 \\ 0 \end{pmatrix}$$

Before we can finish the problem, we need to find the rotation matrix. In the section that follows, we will discuss rotation matrices. For now we will simply quote the result:

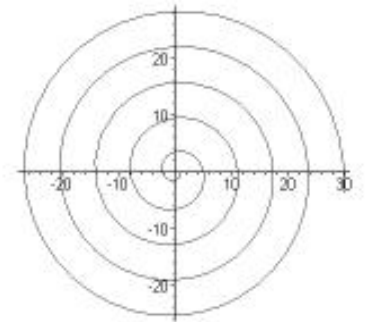
$$\mathbf{R} = \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
 (\hat{x}' \quad \hat{y}' \quad \hat{z}')\vec{r}' &= (\hat{x}' \quad \hat{y}' \quad \hat{z}')\mathbf{R}^{-1} \begin{pmatrix} v_0 t \\ 0 \\ 0 \end{pmatrix} \\
 \vec{r}' &= \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_0 t \\ 0 \\ 0 \end{pmatrix} = v_0 t \begin{pmatrix} \cos \omega t \\ -\sin \omega t \\ 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 (\hat{x}' \quad \hat{y}' \quad \hat{z}')\vec{v}' &= (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{bmatrix} 1 \\ -\omega t \\ 0 \end{bmatrix} v_0 \\
 &= (\hat{x}' \quad \hat{y}' \quad \hat{z}') \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ -\omega t \\ 0 \end{bmatrix} v_0 \\
 \vec{v}' &= v_0 \begin{bmatrix} \cos \omega t - \omega t \sin \omega t \\ -\sin \omega t - \omega t \cos \omega t \\ 0 \end{bmatrix}
 \end{aligned}$$



Position



Velocity

Note that the position and velocity are related by $\vec{v} = \frac{d\vec{r}}{dt}$, as expected.

Notes on Rotation Matrices:

To begin with, we wish to find the rotating unit vectors in terms of the fixed unit vectors.

$$\begin{aligned} \hat{x}' &= \cos \omega t \hat{x} + \sin \omega t \hat{y} \\ \hat{y}' &= -\sin \omega t \hat{x} + \cos \omega t \hat{y} \\ \hat{z}' &= \hat{z} \end{aligned} \quad \begin{array}{l} \text{If the rotation is} \\ \text{about the } z \text{ axis, we} \\ \text{have:} \end{array}$$

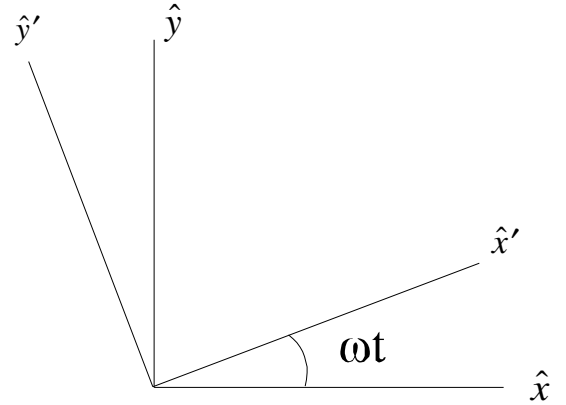


Figure 2 Rotation of basis vectors about the z axis.

Before we proceed to write down the rotation matrix, it is useful to prove an identity. To do this, we use Eq.1 along with the fact that the dot product of two vectors must be the same in both the fixed and the rotating frames.

$$\begin{aligned} (\hat{x}' \quad \hat{y}' \quad \hat{z}') &= (\hat{x} \quad \hat{y} \quad \hat{z})\mathbf{R} \\ \begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} &= \mathbf{R}^T \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \quad \text{since } (\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T \\ (\hat{x}' \quad \hat{y}' \quad \hat{z}') \cdot \begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} &= (\hat{x} \quad \hat{y} \quad \hat{z}) \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \\ (\hat{x} \quad \hat{y} \quad \hat{z})\mathbf{R}\mathbf{R}^T \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} &= (\hat{x} \quad \hat{y} \quad \hat{z}) \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \\ \mathbf{R}\mathbf{R}^T &= \mathbf{I} \Rightarrow \mathbf{R}^{-1} = \mathbf{R}^T \end{aligned}$$

Now we can write the rotation matrix for rotations around the z axis:

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \mathbf{R}^{-1} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \quad \mathbf{R}_z(\omega t) = \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, we can show:

$$\mathbf{R}_x(\omega t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \omega t & -\sin \omega t \\ 0 & \sin \omega t & \cos \omega t \end{pmatrix} \quad \mathbf{R}_y(\omega t) = \begin{pmatrix} \cos \omega t & 0 & \sin \omega t \\ 0 & 1 & 0 \\ -\sin \omega t & 0 & \cos \omega t \end{pmatrix}$$

Now we have established the form of rotation matrices about the three axes, it is useful to think about infinitesimal rotations. For example, let's take a rotation by an angle $\omega_z \Delta t$ about the z axis.

$$\mathbf{R}_z(\omega_z \Delta t) = \begin{pmatrix} \cos \omega_z \Delta t & -\sin \omega_z \Delta t & 0 \\ \sin \omega_z \Delta t & \cos \omega_z \Delta t & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\omega_z \Delta t & 0 \\ \omega_z \Delta t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can generalize this to a sequence of three infinitesimal rotations, one around each axis:

$$\begin{aligned} \mathbf{R}(\omega_x \Delta t, \omega_y \Delta t, \omega_z \Delta t) &= \begin{pmatrix} 1 & -\omega_z \Delta t & 0 \\ \omega_z \Delta t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \omega_y \Delta t \\ 0 & 1 & 0 \\ -\omega_y \Delta t & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\omega_x \Delta t \\ 0 & \omega_x \Delta t & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\omega_z \Delta t & \omega_y \Delta t \\ \omega_z \Delta t & 1 & -\omega_x \Delta t \\ -\omega_y \Delta t & \omega_x \Delta t & 1 \end{pmatrix} \end{aligned}$$

If the rotations are infinitesimal, the order of rotation does not matter. We can then write:

$$\frac{\mathbf{R}(\omega_x \Delta t, \omega_y \Delta t, \omega_z \Delta t) - \mathbf{I}}{\Delta t} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

Thus, any infinitesimal rotation can be considered to be a rotation by an angular velocity ω about an axis given by $\hat{\omega}$.

Now, let's look at how components of vectors in the fixed and rotating frames are related at two times, t_1 and $t_2 = t_1 + \Delta t$.

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \mathbf{R}_2 \begin{pmatrix} x'_2 \\ y'_2 \\ z'_2 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \mathbf{R}_1 \begin{pmatrix} x'_1 \\ y'_1 \\ z'_1 \end{pmatrix},$$

$$\mathbf{R}_2 = \mathbf{R}_1 + \Delta t \dot{\mathbf{R}}_1 = \mathbf{R}_\omega \mathbf{R}_1$$

We can write this last line because we can think of rotation 2 as rotation 1 followed by an infinitesimal rotation about $\hat{\omega}$. (The first rotation is the one nearest the vector upon which it operates, so the first rotation is to the right of the second rotation.)

We can now solve for the infinitesimal rotation:

$$\begin{aligned} \mathbf{R}_\omega \mathbf{R}_1 &= \mathbf{R}_1 + \Delta t \dot{\mathbf{R}}_1, \quad \text{now } \times \text{ by } \mathbf{R}_1^{-1} \\ \mathbf{R}_\omega &= \mathbf{I} + \Delta t \dot{\mathbf{R}}_1 \mathbf{R}_1^{-1} \\ \frac{\mathbf{R}_\omega - \mathbf{I}}{\Delta t} &= \dot{\mathbf{R}}_1 \mathbf{R}_1^{-1} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \\ \dot{\mathbf{R}}_1 \mathbf{R}_1^{-1} \vec{r} &= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \omega_y z - \omega_z y \\ \omega_z x - \omega_x z \\ \omega_x y - \omega_y x \end{pmatrix} = \vec{\omega} \times \vec{r} \end{aligned}$$

We used this result to Prove Eq. 7.

The question then is, in which coordinate system is $\vec{\omega}$ measured? Since we rotate both the unit vectors and the components of matrices, this can get confusing. But we can answer this question by looking at the equation:

$$\mathbf{R}_2 \vec{r}' = \mathbf{R}_\omega \mathbf{R}_1 \vec{r}'$$

We start in the primed system, but then we rotate by \mathbf{R}_1 , which, ignoring the infinitesimal rotation \mathbf{R}_ω puts us in the unprimed system. Hence when \mathbf{R}_ω acts on the system, it is in the fixed coordinate system. That means we need to put $\vec{\omega}$ in the unprimed system when we use the equation

$$\vec{v}' = \vec{v} - \vec{\omega} \times \vec{r}.$$

We can do the same thing, but use the inverse transformation:

$$\begin{pmatrix} x'_2 \\ y'_2 \\ z'_2 \end{pmatrix} = \mathbf{R}_2^{-1} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad \begin{pmatrix} x'_1 \\ y'_1 \\ z'_1 \end{pmatrix} = \mathbf{R}_1^{-1} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},$$

$$\mathbf{R}_2^{-1} = \mathbf{R}_1^{-1} + \Delta t \dot{\mathbf{R}}_1^{-1} = \mathbf{R}_\omega^{-1} \mathbf{R}_1^{-1}$$

We again now solve for the infinitesimal rotation. This time, we are solving for the components of the infinitesimal *inverse* rotation, so we are solving for the components of $-\vec{\omega}$.

$$\mathbf{R}_\omega^{-1} \mathbf{R}_1^{-1} = \mathbf{R}_1^{-1} + \Delta t \dot{\mathbf{R}}_1^{-1}, \quad \text{now } \times \text{ by } \mathbf{R}_1$$

$$\mathbf{R}_\omega^{-1} = \mathbf{I} + \Delta t \dot{\mathbf{R}}_1^{-1} \mathbf{R}_1$$

$$\frac{\mathbf{R}_\omega^{-1} - \mathbf{I}}{\Delta t} = \dot{\mathbf{R}}_1^{-1} \mathbf{R}_1 = \begin{pmatrix} 0 & -(-\omega_z) & (-\omega_y) \\ (-\omega_z) & 0 & -(-\omega_x) \\ -(-\omega_y) & (-\omega_x) & 0 \end{pmatrix}$$

What we really want this time, however, is $\mathbf{R}_1^{-1} \dot{\mathbf{R}}_1$, so we can make use of the relationship:

$$\mathbf{I} = \mathbf{R}^{-1} \mathbf{R}, \quad \dot{\mathbf{I}} = 0 = \dot{\mathbf{R}}^{-1} \mathbf{R} + \mathbf{R}^{-1} \dot{\mathbf{R}}, \quad \dot{\mathbf{R}}^{-1} \mathbf{R} = -\mathbf{R}^{-1} \dot{\mathbf{R}}$$

Then:

$$\mathbf{R}_1^{-1} \dot{\mathbf{R}}_1 \vec{r}' = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \omega_y z' - \omega_z y' \\ \omega_z x' - \omega_x z' \\ \omega_x y' - \omega_y x' \end{pmatrix} = \vec{\omega} \times \vec{r}'$$

This time we have:

$$\mathbf{R}_2^{-1} = \mathbf{R}_\omega^{-1} \mathbf{R}_1^{-1} \vec{r}$$

We start in the unprimed system, but then we rotate by \mathbf{R}_1^{-1} , which puts us in the primed system.

Hence when \mathbf{R}_ω^{-1} acts on the system, it is in the rotating coordinate system. Thus, we need to put $\vec{\omega}$ in the primed system when we use the equation

$$\vec{v} = \vec{v}' + \vec{\omega}' \times \vec{r}'.$$

General Results

Equations 6 and 7 apply specifically to velocities, but in general we can apply the same techniques to other vector quantities as well. This becomes

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \frac{d}{dt} \bar{Q} = \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \left[\frac{d}{dt} \bar{Q}' + \bar{\omega}' \times \bar{Q}' \right]$$

$$\text{where } \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \bar{Q} = \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \bar{Q}'.$$

In particular, we may let Q be the velocities defined by::

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \bar{v} = \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} [\bar{v}' + \bar{\omega}' \times \bar{r}']$$

$$\begin{aligned} \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \frac{d}{dt} \bar{v} &= \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \left[\frac{d}{dt} (\bar{v}' + \bar{\omega}' \times \bar{r}') + \bar{\omega}' \times (\bar{v}' + \bar{\omega}' \times \bar{r}') \right] \\ &= \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \left[\frac{d\bar{v}'}{dt} + \frac{d}{dt} (\bar{\omega}' \times \bar{r}') + \bar{\omega}' \times \bar{v}' + \bar{\omega}' \times (\bar{\omega}' \times \bar{r}') \right] \\ &= \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \left[\frac{d\bar{v}'}{dt} + \frac{d\bar{\omega}'}{dt} \times \bar{r}' + \bar{\omega}' \times \frac{d\bar{r}'}{dt} + \bar{\omega}' \times \bar{v}' + \bar{\omega}' \times (\bar{\omega}' \times \bar{r}') \right] \\ \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \bar{a} &= \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} [\bar{a}' + \dot{\bar{\omega}}' \times \bar{r}' + 2\bar{\omega}' \times \bar{v}' + \bar{\omega}' \times (\bar{\omega}' \times \bar{r}')] \end{aligned}$$

(Eq. 8)

Similarly, it can be shown that:

$$\begin{aligned} \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \frac{d}{dt} \bar{v}' &= \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \left[\frac{d}{dt} (\bar{v} - \bar{\omega} \times \bar{r}) - \bar{\omega} \times (\bar{v} - \bar{\omega} \times \bar{r}) \right] \\ &= \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \left[\frac{d\bar{v}}{dt} - \frac{d}{dt} (\bar{\omega} \times \bar{r}) - \bar{\omega} \times \bar{v} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) \right] \\ &= \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \left[\frac{d\bar{v}'}{dt} - \frac{d\bar{\omega}}{dt} \times \bar{r} - \bar{\omega} \times \frac{d\bar{r}}{dt} - \bar{\omega} \times \bar{v}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}) \right] \\ \begin{pmatrix} \hat{x}' & \hat{y}' & \hat{z}' \end{pmatrix} \bar{a}' &= \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix} [\bar{a} - \dot{\bar{\omega}} \times \bar{r} - 2\bar{\omega} \times \bar{v} + \bar{\omega} \times (\bar{\omega} \times \bar{r})] \end{aligned}$$

(Eq. 9)

Example 2:

Returning to Example 1, we wish now to calculate the effective acceleration experienced by the block on the turntable.

$$\begin{aligned}(\hat{x}' \quad \hat{y}' \quad \hat{z}')\vec{a}' &= [-2\omega v_0 \hat{z} \times \hat{x} + \omega^2 v_0 t \hat{z} \times (\hat{z} \times \hat{x})] \\&= -2\omega v_0 \hat{y} - \omega^2 v_0 t \hat{x} \\&= (\hat{x} \quad \hat{y} \quad \hat{z}) \begin{pmatrix} -\omega^2 v_0 t \\ -2\omega v_0 \\ 0 \end{pmatrix} \\&= (\hat{x}' \quad \hat{y}' \quad \hat{z}') \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -\omega^2 v_0 t \\ -2\omega v_0 \\ 0 \end{pmatrix} \\ \vec{a}' &= \begin{pmatrix} -\omega^2 v_0 t \cos \omega t - 2\omega v_0 \sin \omega t \\ \omega^2 v_0 t \sin \omega t - 2\omega v_0 \cos \omega t \\ 0 \end{pmatrix}\end{aligned}$$