Scattering Kinematics in the Lab and Center of Mass

Let us take a beam of particles of known kinetic energy incident on a stationary target in the lab. We take the incident direction of the beam to be the *z* direction and assume that the incident particle scatters in the *x*-*z* plane. Let us also assume that we know the masses of the beam and target particles and the energy loss (excitation energy), if any.

The equations that relate all the energies and momenta in the lab are as follows:

Lab:

$$\begin{bmatrix} 0 \\ 0 \\ p_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1 \sin\theta_1 \\ 0 \\ p_1 \cos\theta_1 \end{bmatrix} + \begin{bmatrix} p_2 \sin\theta_2 \\ 0 \\ p_2 \cos\theta_2 \end{bmatrix}$$

$$t_0 + 0 = t_1 + t_2 + E_{ex}$$

$$p_0^2 = 2m_1 t_0, \quad p_1^2 = 2m_1 t_1, \quad p_2^2 = 2m_2 t_2$$

If we are given t_0, we can find p_0. In principle, if we specify θ_1 (or any other variable such as θ_2, t_1, etc.) then we can find all the other unknowns. In practice, it often takes quite a bit of algebra. Since we tend to define everything in terms of the center of mass (cm, or zero momentum) frame anyway, it is easiest to go to the cm frame at this point and solve the problem there. Let's use upper case letters for cm quantities and lower case letters for lab quantities. Then, if we call the velocity of the cm frame with respect to the lab *v*, the general equation for transformation of velocities is:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

Then the conservation of momentum equation in the center of mass can be transformed easily by writing everything in terms of velocities:

$$m_1 \begin{bmatrix} 0 \\ 0 \\ v_0 - v \end{bmatrix} + m_2 \begin{bmatrix} 0 \\ 0 \\ -v \end{bmatrix} = m_1 \begin{bmatrix} v_1 \sin\theta_1 \\ 0 \\ v_1 \cos\theta_1 - v \end{bmatrix} + m_2 \begin{bmatrix} v_2 \sin\theta_2 \\ 0 \\ v_2 \cos\theta_2 - v \end{bmatrix}$$

Since it is easier to use this in terms of momentum, we rewrite it as:

$$\begin{bmatrix} 0 \\ 0 \\ p_0 - m_1 v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -m_2 v \end{bmatrix} = \begin{bmatrix} p_1 \sin\theta_1 \\ 0 \\ p_1 \cos\theta_1 - m_1 v \end{bmatrix} + \begin{bmatrix} p_2 \sin\theta_2 \\ 0 \\ p_2 \cos\theta_2 - m_2 v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first thing to do is find the value of v that gives us zero momentum. This is easily solved from the initial momenta:

$$m_1 v_0 - m_1 v - m_2 v = 0$$

$$\Rightarrow v = \frac{m_1}{m_1 + m_2} v_0 = \frac{\mu}{m_2} v_0$$

where we have made use of the fact that $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$.

Now, in terms of cm variables, we can also write the momentum and energy relations as:

CM:

$$\begin{bmatrix} 0 \\ 0 \\ P_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -P_0 \end{bmatrix} = \begin{bmatrix} P\sin\Theta \\ 0 \\ P\cos\Theta \end{bmatrix} + \begin{bmatrix} -P\sin\Theta \\ 0 \\ -P\cos\Theta \end{bmatrix}$$

$$T_{01} + T_{02} = T_1 + T_2 + E_{ex}$$

$$P_0^2 = 2m_1 T_{01} = 2m_2 T_{02}, \quad P^2 = 2m_1 T_1 = 2m_2 T_2$$

We can combine the last two equations to get useful expressions for cm momenta in terms of the cm energies:

$$T_{01} + T_{02} = \frac{P_0^2}{2m_1} + \frac{P_0^2}{2m_2} = \frac{P_0^2}{2\mu}$$

$$\Rightarrow P_0 = \sqrt{2\mu(T_{01} + T_{02})}$$

and similarly

$$P = \sqrt{2\mu(T_1 + T_2)} = \sqrt{2\mu(T_{01} + T_{02} - E_{ex})}$$

Finally, we need to connect the lab quantities to the cm quantities. This is done by relating the two expressions for momentum conservation in the cm. Before the collision, we have:

$$P_0 = p_0 - m_1 v = m_2 v = \mu v_0$$

$$T_{01} = \frac{P_0^2}{2m_1}, \quad T_{02} = \frac{P_0^2}{2m_2}$$

While after the collision, we similarly have:

$$P = \sqrt{2\mu(T_{01} + T_{02} - E_{ex})}$$
$$T_1 = \frac{P^2}{2m_1}, \quad T_2 = \frac{P^2}{2m_2}$$

Now we know the momenta and energies in the cm frame. Where we go from here depends on what information we are give. For example, if we know the cm scattering angle, we can go back and find the momenta in the lab frame. Squaring the *x* and *z* components of the lab momentum and adding these (to eliminate the lab angles), we get:

$$p_1^2 = p_1^2\cos^2\theta_1 + p_1^2\sin^2\theta_1 = (P\sin\Theta)^2 + (P\cos\Theta + m_1 v)^2$$
$$p_2^2 = p_2^2\cos^2\theta_2 + p_2^2\sin^2\theta_2 = (-P\sin\Theta)^2 + (-P\cos\Theta + m_2 v)^2$$
$$\Rightarrow t_1 = \frac{p_1^2}{2m_1}, \quad t_2 = \frac{p_2^2}{2m_2}$$

And we can find the lab scattering angles by dividing the *x* components of momentum by the *z* components:

$$\tan\theta_1 = \frac{P\sin\Theta}{P\cos\Theta + m_1 v}, \quad \tan\theta_2 = \frac{-P\sin\Theta}{-P\cos\Theta + m_2 v}$$

On the other hand, if we know the lab scattering angle, it is a little more complicated:

$$P^2 = (p_1\sin\theta_1)^2 + (p_1\cos\theta_1 - m_1 v)^2$$
$$\Rightarrow p_1^2 - 2p_1 m_1 v\cos\theta_1 + \left(m_1^2 v^2 - P^2\right) = 0$$

We can solve this quadratic equation for p_1. Then we can obtain the other lab quantities by using the relations:

$$\Rightarrow t_1 = \frac{p_1^2}{2m_1}$$
$$\Rightarrow t_2 = t_0 - t_1 - E_{ex} \quad \Rightarrow p_2 = \sqrt{2m_2 t_2}$$
$$\Rightarrow p_2\sin\theta_2 = -p_1\sin\theta_1$$

And the cm scattering angle can be obtained by reversing the process that gave us θ_1 above:

$$\tan\Theta = \frac{p_1\sin\theta_1}{p_1\cos\theta_1 - m_1 v}$$