1. Two positive charges (+Q) are placed in fixed locations a distance $y_0$ from the axis of a tube and at distances of $+x_0$ and $-x_0$ from the center of the tube. Take the center of the tube to be the origin of the coordinate system. A negative charge (−q, take q to be the magnitude of the charge and hence positive) is placed inside the tube. The negative charge moves without friction in the tube.

(a – 5 points) Write an expression for the potential energy of the negative charge as a function of $x$, the distance from the origin to the negative charge. (Consider $x_0$ and $y_0$ to be positive, but $x$ to be either positive or negative. Just in case you don’t remember, the formula for the Coulomb force is $F=kq_1q_2/r^2$.)

$$U(x) = -\frac{kQq}{\sqrt{(x_0 + x)^2 + y_0^2}} - \frac{kQq}{\sqrt{(x_0 - x)^2 + y_0^2}}$$
(b – 5 points) The charge is placed in a uniform electric field, $E$, that points to the right. What is the potential energy of the negative charge associated with the electric field? ($F = qE$.) Take this potential energy to have a value of 0 when the negative charge is at the origin.

$$U_E = qx$$

(c – 5 points) In the graph below, the potential energy wells with and without the uniform electric field are illustrated.

The negative charge is released from rest just barely to the right of the origin when there is no uniform electric field applied. Using these curves, describe details (give numbers) of the motion of the subsequent motion of the negative charge.

*Oscillates between the origin and ~0.33 m.*
*Reaches a maximum T at 0.22 m.*

(d – 5 points) Repeat part (c) for the case when the electric field is applied and the negative charge is again released from rest just barely to the right of the origin.

*Oscillates between the origin and ~ −0.38 m.*
*Reaches a maximum T at −0.23 m.*
2. The moment of inertia of a lamina is given by the formula
\[ I = \int d^2 \sigma dA \]
where \( \sigma \) is the mass per unit area of the lamina. (Recall that a lamina is a thin, flat sheet of material.)

(a – 10 points) A square lamina of length \( a \) on a side and mass \( m \) is suspended by one corner to make a physical pendulum. The density of the lamina is constant. What is its frequency of oscillation? (Hint: If you get stuck, try writing a differential equation in terms of \( \theta \).) Give your answer in terms of \( m, a, \) and \( g \) (gravitational acceleration).

\[ l = \frac{m}{a^2} \int_0^a \int_0^a (x^2 + y^2) \, dx \, dy = \frac{2}{3} ma^2 \quad l = \frac{\sqrt{2}}{2} a \]

\[ \omega = \sqrt{\frac{mgl}{I}} = \sqrt{\frac{3\sqrt{2}g}{4a}} \]

(b – 5 points) The same square is suspended by the midpoint of one side. What is its frequency of oscillation now?

\[ l = \frac{m}{a^2} \int_0^{a/2} \int_{-a/2}^{a/2} (x^2 + y^2) \, dx \, dy = \frac{5}{12} ma^2 \quad l = \frac{1}{2} a \]

\[ \omega = \sqrt{\frac{mgl}{I}} = \sqrt{\frac{6g}{5a}} \]

3. A driven series LRC circuit is a damped, driven oscillator that satisfies the differential equation:
\[ \frac{q}{C} - R \frac{dq}{dt} + L \frac{d^2q}{dt^2} = V_0 \cos \omega t \]
where \( q \) is the charge on the capacitor, \( C \) is capacitance, \( R \) is resistance, \( L \) is inductance, \( V_0 \) is the voltage of the AC power supply, and \( \omega \) is the angular frequency of the power supply. (Don’t worry, you really don’t need to remember AC circuits at all to do this problem. You just need the equation.) Express all of your answers in terms of \( C, L, R, \) and \( \omega \). Assume \( Q \) for the circuit is very large.

(a – 5 points) What are the values of \( \omega_0 \) and \( \beta \)?

\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad \beta = \frac{R}{2L} \]

(b – 5 points) What is the resonant frequency (the frequency where the amplitude of \( q(t) \) is maximized)?

\[ A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2} \]
is maximized when \( (\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2 \) is minimized.
\[
\begin{align*}
\frac{d}{d\omega}(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2 &= 2(\omega_0^2 - \omega^2)(-2\omega) + 8\beta^2\omega = 0 \\
-(\omega_0^2 - \omega^2) + 2\beta^2 &= 0 \\
\omega &= \sqrt{\omega_0^2 - 2\beta^2} = \frac{1}{\sqrt{LC}} \frac{R^2}{2L^2}
\end{align*}
\]

(c – 5 points) What is the value of Q for this oscillator?

\[
Q = \frac{\omega_0}{2\beta} = \frac{L}{R} \frac{1}{\sqrt{LC}} = \frac{1}{R} \sqrt{\frac{L}{C}}
\]

(d – 5 points) Describe in words what Q means for this oscillator.

I would accept
1) sharpness of the response function
2) natural \(\omega/\text{FWHM}\)
3) energy/energy lost per cycle

4. A basketball of radius \(R\) and moment of inertia \(I\) rolls without slipping down an inclined plane that makes an angle \(\theta\) with respect to the horizontal.

(a – 5 points) Sketch the coordinate system you are going to use. Show the origin and direction of increasing value for each linear coordinate.

(b – 5 points) Find the acceleration of the ball by using forces and torques.

\[
\begin{align*}
mx &= mg \sin \theta - F_f \\
l\ddot{\varphi} &= RF_f
\end{align*}
\]
\[ \ddot{x} = R^2 F_f = \frac{R^2}{I} (mg \sin \theta - m \ddot{x}) \]
\[ \left(1 + \frac{mR^2}{I}\right) \ddot{x} = \frac{mR^2}{I} g \sin \theta \]
\[ \ddot{x} = \frac{mR^2}{I + mR^2} g \sin \theta \]

(c – 5 points) Find the kinetic and potential energies of the ball.

\[ T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} l \dot{\phi}^2 = \frac{1}{2} \left( m + \frac{l}{R^2} \right) \dot{x}^2 \]
\[ U = mgy = -mgx \sin \theta \]

(d – 5 points) Find an equation of motion by using one of the two methods of obtaining a differential equation with conservation of energy. (You can call the total energy \( E \).)

\[ \frac{1}{2} \left( m + \frac{l}{R^2} \right) \dot{x}^2 - mgx \sin \theta = E \]

(e – 5 points) Find an equation of motion by using the second method of obtaining a differential equation with conservation of energy.

\[ \left( m + \frac{l}{R^2} \right) \ddot{x} - mg \dot{x} \sin \theta = 0 \]
\[ \left( m + \frac{l}{R^2} \right) \dot{x} - mg \sin \theta = 0 \]

(f – 5 points) Using the simplest of the equations in (c) and (d), find the acceleration of ball.

\[ \ddot{x} = \frac{mR^2}{I + mR^2} g \sin \theta \]

5. A proton and an antiproton collide and annihilate each other producing new particles. (An antiproton has the same mass as a proton, but a negative charge.) The kinetic energy of each particle is \( T_0 \) when the particles are far from each other.

(a – 5 points) If two particles of equal mass are created in the process, what is the maximum mass they could each have? (Call the proton mass \( m \) and ignore any relativistic kinematic effects.)

*If all the kinetic energy and mass were converted to the mass of two particles at rest, the total mass would be \( T_0/c^2 + m \).*
(b – 10 points) How does this answer differ if an antiproton of kinetic energy \(2T_0\) strikes a proton at rest? (There are several ways you could approach this problem. The most obvious way is a little harder than some. If you do it this way, make the approximation that \(T_0^2 \ll m^2c^4\) in the square root to simplify things.)

The easy way – put in cm frame. There the initial momentum and velocity are

\[
p_0 = \sqrt{2m2T_0} \quad v_0 = 2 \sqrt{\frac{T_0}{m}}.
\]

The velocity of the cm is half this, so the velocity in the cm is also half this:

\[
p_{cm} = \frac{T_0}{\sqrt{m}} \quad p_{cm} = \sqrt{mT_0}
\]

and the kinetic energy is

\[
T = \frac{p_{cm}^2}{2m} = \frac{T_0}{2}
\]

and the maximum mass is

\[
m_{max} = m + \frac{T_0}{2c^2}
\]

The harder way is to conservation of energy and momentum in the lab.

\[
2mc^2 + 2T_0 = 2m_{max}c^2 + 2T_f
\]

\[
p_0 = 2\sqrt{mT_0} = 2p_f = 2 \sqrt{2m_{max}T_f} \quad \rightarrow \quad 4mT_0 = 8m_{max}T_f
\]

\[
mc^2 + T_0 = m_{max}c^2 + T_f \quad \rightarrow \quad mc^2 + T_0 = m_{max}c^2 + \frac{m}{2m_{max}}T_0
\]

\[
m_{max} = \frac{2(mc^2 + T_0) \pm \sqrt{4(mc^2 + T_0)^2 - 8mc^2T_0}}{4c^2} = \frac{2(mc^2 + T_0) \pm 2\sqrt{m^2c^4 + T_0^2}}{4c^2}
\]

\[
m_{max} \approx \frac{2(mc^2 + T_0) + 2mc^2}{4c^2} = m + \frac{T_0}{2c^2}
\]

6. (10 points) Write down the three equations for the motion of a baseball in midair in three dimensions. Include linear and quadratic drag forces. Take the vertical direction to be the \(z\) direction and assume a wind is blowing in the \(y\) direction. (Use \(c_1\) and \(c_2\) for the linear and quadratic drag coefficients. These are not functions of the ball diameter.)

\[
m\ddot{x} = -c_1\dot{x} - c_2\sqrt{\dot{x}^2 + (\dot{y} - v_w)^2 + \dot{z}^2}\dot{x}
\]

\[
m\ddot{y} = -c_1(\dot{y} - v_w) - c_2\sqrt{\dot{x}^2 + (\dot{y} - v_w)^2 + \dot{z}^2}(\dot{y} - v_w)
\]

\[
m\ddot{z} = -c_1\dot{z} - c_2\sqrt{\dot{x}^2 + (\dot{y} - v_w)^2 + \dot{z}^2}\dot{z} - mg
\]