Instructor: Lawrence Rees

CID _________________ (Please do not include your name.)

There are seven questions each worth 15 points.

1. A block of mass $M$ slides frictionlessly on an air table. It is constrained to move on the air table in one dimension only. On top of the block is a post that supports a pendulum of length $d$ and mass $m$. The block is connected to the wall by a spring of spring constant $k$. Consider the spring, post, and pendulum rod to be massless.

As drawn there are four variables in the problem: $X$, $x$, $y$, and $\theta$.

(a – 5 points) How many independent variables are there in this problem? Choose an appropriate set of variables for the problem. Write any equations of constraint you would use.

Two. We will choose $X$ and $\theta$.

\[
\begin{align*}
\dot{y} &= d - d \cos \theta \\
\dot{x} &= d \sin \theta
\end{align*}
\]

Other options are also acceptable, but probably few will be used.

(b – 10 points) Describe each term you need to include in the Lagrangian and indicate the variable or variables upon which this term depends. Write your answer in terms of the variables you chose in part (a), their time derivatives, conjugate momenta, etc. Just be sure that the variables you use are appropriate for the Lagrangian. (You should actually work out the Lagrangian to be sure if you’re correct; however, we will not grade you on the entire Lagrangian.)

(Note that the number of lines in the table is not necessarily an indication of the number of energy terms you need to include.)

<table>
<thead>
<tr>
<th>Description of Energy Term in the Lagrangian</th>
<th>Depends on</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kinetic energy of the block</td>
<td>$\dot{X}$</td>
</tr>
<tr>
<td>Kinetic energy of the pendulum</td>
<td>$\dot{X}, \dot{\theta}, \theta$</td>
</tr>
<tr>
<td>Gravitational potential energy of the pendulum</td>
<td>$\theta$</td>
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<td>Potential energy of the spring</td>
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Some details:

\[ T = \frac{1}{2} M \ddot{x}^2 + \frac{1}{2} m (\dot{x} + \dot{\theta})^2 + \frac{1}{2} m \dot{\theta}^2 \]

\[ = \frac{1}{2} (M + m) \dot{x}^2 + \frac{1}{2} m \dot{\theta}^2 + m d \cos \theta \dot{\theta} \]

\[ U = mgd (1 - \cos \theta) + \frac{1}{2} k (X - X_0)^2 \]

\[ p_x = (M + m) \dot{x} + m d \cos \theta \dot{\theta} \]

\[ p_{\theta} = m d^2 \dot{\theta} + m d \cos \theta \dot{X} \]
2. (a – 10 points) How would you work this same problem using a Hamiltonian formalism? Be sure to write Hamilton’s Equations of motion for the last step.

A minimum answer would be:

• Find Lagrangian for the system (in terms of angle and its time derivative).
• Find momentum conjugate to the angle.
• Solve for the time derivative of the angle in terms of the conjugate momentum,

At this point there are two options:
1. • Find T in terms of the conjugate momentum.
   • $H = T + U$

2. • Find L, $\mathcal{H} = \mathcal{P} \dot{\theta} - L$
   • Express H in terms of the conjugate momentum.

Then:

$$\frac{\partial H}{\partial \dot{\theta}} = -\dot{p}, \quad \frac{\partial H}{\partial p} = \dot{\theta}$$

(b – 5 points) Reproduce the table of Problem 1, but indicate the variables you need to use for the Hamiltonian. (You can use standard symbols or word descriptions, you need not actually solve for the variables. Be careful to think about how variables are related. There are no magical simplifications that you would find by working through the problem in detail.)

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Don’t take off points again here if they missed $\theta$ in the K term for the pendulum – just be sure that they understand that $\dot{X}$ and $\dot{\theta}$ both depend on both conjugate momenta as well as on $\theta$. 
3. A proton scatters from a calcium nucleus via the Coulomb potential. (Note that both the proton and the calcium nucleus have positive charges. Ignore electrons in the calcium atom.) Recall that

\[ U_{\text{cent}} = \frac{L^2}{2 \mu r^2} \]

(a – 5 points) At very large \( r \), which potential is larger, the Coulomb potential or the Centrifugal potential? How do you know?

*The Coulomb potential as it goes as 1/r whereas the Centrifugal potential goes as 1/r².*

(b – 10 points) Sketch the centrifugal potential, the Coulomb potential, and the total effective potential.

*Don’t worry about where the Coulomb potential gets smaller than the Coulomb potential. But be sure that the signs of both potentials are correct and that there are no potential minima for bound states to be present.*
4. You are sitting in a rotating room. The room is turning so that you face inward and are always moving to your right. In what direction is the centrifugal force? If you lean forward, what is the direction of the Coriolis force? As the room comes to a stop, what is the direction of the transverse force?

The centrifugal force is outward from the center of the room.
If you lean forward, the Coriolis force is to right.
The transverse force is also to the right.

\[ m\ddot{r} = \vec{F}_{\text{real}} + m\dot{r} \times \vec{\Omega} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} \]
5. (a – 15 points) In the northern hemisphere, do winds around a low pressure area circulate clockwise or counter-clockwise (as viewed from above the atmosphere, as in a weather map)? Explain your answer in terms of the Coriolis force.

In a low pressure area, the winds tend to blow in toward the low pressure zone. As they do so, they experience a force toward the right (the $\hat{v} \times \vec{\Omega}$ direction). This leads to a counterclockwise flow.

(b – 5 points) Is there ever a vertical (with respect to the surface of the earth) component of the Coriolis force on moving air masses? Briefly explain.

Since the angular velocity is directed toward Polaris, and is not vertically upward, there will be upward and downward components of the Coriolis force (whenever the wind has an eastward or westward component.)

6. A binary star system consists of one star that is twice as massive as the other star. Describe the motion of the stars. Sketch their orbits.

The important points are:
the orbits are ellipses of the same eccentricity
the center of mass is a focus for each ellipse
the larger star is always opposite the smaller star and twice as close to the center of mass

If you can tell from the words, and or sketch, that they understand the concept, they don’t have to have all the details I’ve written.

7. What is the meaning of “ignorable coordinates”? If a coordinate is ignorable, what important consequence follows from Lagarange’s equations of motion?

$q$ is an ignorable coordinate if it doesn’t explicitly appear in the lagrangian. In this case the momentum conjugate to $q$ will be a conserved quantity.