

Physics 318 – Review for Final Exam

Final Exam

There will be five problems. I'm giving you some hints so you can think about the "tricks" before you get there. That way I'll test you more on the core concepts. So be sure to review the basic material thoroughly as well as review the specifics noted below.

1) A one dimension wave or heat transfer problem. Either the equation or the boundary conditions will be nonhomogeneous.

– Nonhomogeneous boundary conditions: Solve the equation with homogeneous boundary conditions and add a particular solution that arises when u_t or $u_{tt} = 0$ and the boundary conditions are nonhomogeneous.

– Nonhomogeneous equation: Solve the homogeneous equation in x and then make the expansion coefficients functions of t . Expand the inhomogeneous equation in terms of the same basis functions. You will get an ordinary differential equation in t for the coefficients.

2) Laplace's Equation in polar, cylindrical, or spherical coordinates. Review sections 4.4, 4.5, 5.1, and 5.2 and review the special functions that arise in these problems.

3) Sturm–Liouville problem. Memorize the Sturm-Liouville form. Practice putting equations in Sturm–Liouville form and interpreting the results.

Hint: Write equation in the form $y'' + f(x)y' + \dots$, then multiply everything by $p(x)$. Since Sturm–Liouville form is $py'' + p'y' + \dots$, you know $p'(x) = p(x)f(x)$. This can be

integrated out: $\int \frac{dp}{p} = \int f(x)dx$ to find $p(x)$.

4) A heat-transfer equation on a semi-infinite ($0 \leq x \leq \infty$) rod. This will involve Laplace transforms and convolutions. The table of Laplace transforms on the next page will be provided.

5) A problem involving Fourier transforms. You will NOT be given the equations for the Fourier transform or inverse transform. Please memorize these with the book's choice of phase ($\sqrt{2\pi}$ in the denominator of both the transform and the inverse transform.) If the resulting integrals are simple enough, you may be asked to evaluate them.

Information that will be given

- Calculator are permitted.
- There is a three-hour time limit. There are no exceptions to this unless you have medical problems.
- Be sure to show your work, as we can only give credit for what is on your paper. Whenever there is a separation constant (such as k_n) that takes on specific values, be sure to tell me what values it can have.
- Be sure your copy of the test has five problems.
- Please sign the roll and list your test number, just in case there are problems with your CID.
- Write your CID on each sheet, as we will split up the pages for grading.

Laplace Transforms:

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(f'(t)) = sL(f(t)) - f(0)$$

$$L(f''(t)) = s[sL(f(t)) - f(0)] - f'(0)$$

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} L(f(t))(s)$$

$$L(e^{\alpha t} f(t)) = F(s - \alpha) \text{ where } F(s) = L(f(t))(s)$$

$$L(H(t-a)f(t-a))(s) = e^{-as} F(s)$$

$$L(f * g) = L(f)L(g), \quad f * g = \int_0^t f(t-\tau)g(\tau)d\tau$$

$$L\left(\frac{a}{2s\sqrt{t}} e^{-a^2/(4t)}\right) = e^{-a\sqrt{s}}$$

$$L\left(\frac{3/2}{s^2 + k^2}\right), \quad L(\sin kt) = \frac{k}{s^2 + k^2}$$

Orthogonal functions:

1. Sines and cosines I.	
functions	$X(x) = A \cos \mu_n x + B \sin \mu_n x$
diff. eq.	$X''(x) = -\mu_n X(x)$
Range	0 to L
orthogonality	$\int_0^L \sin \mu_n x \sin \mu_m x dx = \frac{L}{2} \delta_{mn}$, $\int_0^L \cos \mu_n x \cos \mu_m x dx = \frac{L}{2} \delta_{mn}$, $\int_0^L \sin \mu_n x \cos \mu_m x dx = 0$
orthonormal functions	$\varphi_n(x) = \sqrt{\frac{2}{L}} \sin \mu_n x$, $\sqrt{\frac{2}{L}} \cos \mu_n x$, weighting function = 1

2. Sines and cosines II.	
functions	$X(x) = A \cos \mu_n x + B \sin \mu_n x$
diff. eq.	$X''(x) = -\mu_n X(x)$
Range	-L to L
orthogonality	$\int_0^L \sin \mu_n x \sin \mu_m x dx = L \delta_{mn}$, $\int_0^L \cos \mu_n x \cos \mu_m x dx = L \delta_{mn}$, $\int_0^L \sin \mu_n x \cos \mu_m x dx = 0$
orthonormal functions	$\varphi_n(x) = \sqrt{\frac{1}{L}} \sin \mu_n x$, $\sqrt{\frac{1}{L}} \cos \mu_n x$, weighting function = 1

3. Sines and cosines III.	
functions	$\Theta(\theta) = A \cos m\theta + B \sin m\theta$
diff. eq.	$\Theta''(x) = -\mu_n \Theta(\theta)$
Range	0 to 2π
orthogonality	$\int_0^{2\pi} \sin mx \sin nx dx = \pi \delta_{mn}$, $\int_0^{2\pi} \cos mx \cos nx dx = \pi \delta_{mn}$, $\int_0^{2\pi} \sin mx \cos nx dx = 0$
orthonormal functions	$\varphi_m(\theta) = \sqrt{\frac{1}{\pi}} \sin m\theta$, $\sqrt{\frac{1}{\pi}} \cos m\theta$, weighting function = 1

4. Bessel functions	
functions	$R(r) = AJ_m(\lambda_{mn}r) + BY_m(\lambda_{mn}r)$
diff. eq.	$r^2 R'' + rR' + (\lambda_{mn}^2 r^2 - m^2)R = 0$
λ_{mn}	$\lambda_{mn} = \frac{\alpha_{mn}}{a}$
range	0 to a
orthogonality	$\int_0^a J_m(\lambda_{mn}r)J_m(\lambda_{m'n'}r)rdr = \frac{a^2}{2} J_{m+1}^2(\alpha_{mn})\delta_{nn'}$
orthonormal functions	$\varphi_{mn}(r) = \frac{\sqrt{2}}{aJ_{m+1}(\alpha_{mn})} J_m(\lambda_{mn}r)$, weighting function = r

5. Euler's equation (cylindrical coordinates)	
functions	$R(r) = A + b \ln r, m = 0; R(r) = A\left(\frac{r}{a}\right)^m + B\left(\frac{a}{r}\right)^m, m > 0$
diff. eq.	$r^2 R'' + rR' - m^2 R = 0$
Range	0 to ∞

6. Euler's equation (spherical coordinates)	
functions	$R(r) = A\left(\frac{r}{a}\right)^\ell + B\left(\frac{a}{r}\right)^{\ell+1}$
diff. eq.	$r^2 R'' + 2rR' - \ell(\ell+1)R = 0$
Range	0 to ∞

7. Associated Legendre polynomials (Or if $\ell = 0$, regular Legendre polynomials)	
functions	$P_\ell^m(\cos \theta)$
diff. eq.	$\theta'' + \cot \theta \theta' + [\ell(\ell+1) - m^2 \csc^2 \theta]\theta = 0$
Range	0 to π
orthogonality	$\int_0^\pi P_\ell^m(\cos \theta)P_{\ell'}^m(\cos \theta) \sin \theta d\theta = \delta_{\ell\ell'} \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}$

8. Spherical harmonics	
functions	$Y_\ell^m(\theta, \phi)$
diff. eq.	$Y_{\theta\theta} + \cot \theta Y_\theta + \csc^2 \theta Y_{\phi\phi} + \ell(\ell + 1)Y = 0$
Range	$0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$
orthogonality	$\int_0^{2\pi} \int_0^\pi Y_\ell^m(\theta, \phi) \overline{Y_{\ell'}^{m'}(\theta, \phi)} \sin \theta d\theta d\phi = \delta_{\ell\ell'} \delta_{mm'}$ weighting function = sin(θ)
useful info	$Y_\ell^m(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\phi}$

9. Spherical Bessel and Neumann functions	
functions	$R(r) = Aj_\ell(kr) + Bn_\ell(kr)$
diff. eq.	$r^2 R'' + 2rR' + (k^2 r^2 - \ell(\ell + 1))R = 0$
range	0 to a (or 0 to ∞ , depending on the problem)
orthogonality	$\int_0^a j_\ell\left(\frac{\alpha_m r}{a}\right) j_\ell\left(\frac{\alpha_{n'} r}{a}\right) r^2 dr = \frac{a^3}{2} [j_{\ell+1}(\alpha_{\ell n})]^2 \delta_{nn'}$ useful when b.c. are $R(a) = 0$.

Be able to:

1. Separate equations and be able to find the equations for each of the separation variables.
2. Use the table of functions given in Review 2. Apply boundary conditions, including finiteness constraints, to give allowed values of constants, etc.
3. Be able to solve the heat equation in two-dimensions (Cartesian only) with a source and with non-homogeneous boundary conditions.
4. Prove for the Sturm-Liouville problem that eigenvalues are real and that the eigenfunctions corresponding to distinct eigenvalues are orthogonal.
5. Transform an equation into Sturm-Liouville form and determine what the orthogonality relation must be for the eigenfunctions.
6. Be able to solve the wave and heat equations on the ranges $-\infty < x < \infty$ and $0 \leq x < \infty$.
7. Use Fourier transforms to find the power spectrum (the absolute square of the Fourier transform) of a wave in the time domain.
8. Using tables and sets of rules, be able to solve ordinary differential equations using Laplace Transforms.
9. Know the integral equations for Fourier and Laplace Transforms and for Inverse Fourier Transforms.
10. Understand how to use Heavide functions and Dirac Delta functions. For heavide functions: know the definition and be able to construct step, square, triangular, and ramping functions from them. Be able to solve integrals that include (one-dimensional) Dirac delta functions.

Solve the following problems:

1. The static electric potential satisfies the relationship $\nabla^2 V = 0$ in regions of space where no electric charge is located. A conducting cube of side 30 cm has an electric potential of +70 V everywhere on the surface. Find the electric potential inside the cube. Take the origin of your coordinate system to be in the center of the cube. (No I won't give you a problem this long!) (*Important ideas: Separation of variables in three dimensions, non-homogeneous boundary conditions and sum of linear solutions, using orthogonality conditions.*)

You may recognize that by Gauss's Law the electric field inside the cube must be zero, so the potential is a constant 70 V everywhere inside. But we need to work the problem the hard way.

We'll need to break the problem into simpler problems. Let $a = 15$ cm and $V_0 = +70$ V. Then we start with:

$$\nabla^2 u_1 = 0, \quad u_1(x, \pm a, z) = 0, \quad u_1(x, y, \pm a) = 0, \quad u_1(-a, y, z) = 0, \quad u_1(a, y, z) = V_0$$

First, we do separation of variables:

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

We're going to need some positive separation constants and some negative. Zero may also be an option. Let's start with Y and Z :

$$Y'' = -k_y^2 Y$$

$$Y = A \cos k_y y + B \sin k_y y$$

$$Y(\pm a) = 0 \Rightarrow$$

$$A \cos k_y a \pm B \sin k_y a = 0$$

$$B = 0$$

$$k_y = \frac{(2m-1)\pi}{2a}, \quad m = 1, 2, 3, \dots$$

$$Y(y) \propto \cos k_y y$$

$$Y'' = 0$$

$$Y = Ay + B$$

$$Y(\pm a) = 0 \Rightarrow$$

$$Y = 0$$

$$Y'' = +k_y^2 Y$$

$$Y = A \cosh k_y y + B \sinh k_y y$$

$$Y(\pm a) = 0 \Rightarrow$$

$$Y = 0$$

$$Z'' = -k_z^2 Z$$

$$Z = A \cos k_z z + B \sin k_z z$$

$$Z(\pm a) = 0 \Rightarrow$$

$$A \cos k_z a \pm B \sin k_z a = 0$$

$$B = 0$$

$$k_z = \frac{(2n-1)\pi}{2a}, \quad n = 1, 2, 3, \dots$$

$$Z(z) \propto \cos k_z z$$

$$Z'' = 0$$

$$Z = Ay + B$$

$$Z(\pm a) = 0 \Rightarrow$$

$$Z = 0$$

$$Z'' = +k_z^2 Z$$

$$Z = A \cosh k_z z + B \sinh k_z z$$

$$Z(\pm a) = 0 \Rightarrow$$

$$Z = 0$$

We see that the separation constant for X must be positive if the other two are negative.

$$X'' = +k_x^2 X$$

$$X = A \cosh k_x x + B \sinh k_x x$$

$$X(-a) = 0 \Rightarrow$$

$$A \cosh k_x a - B \sinh k_x a = 0$$

$$B = A \coth k_x a$$

$$X(x) \propto \cosh k_x x + \coth k_x a \sinh k_x x$$

$$k_x^2 = k_y^2 + k_z^2$$

$$k_x = \frac{\pi}{2a} \sqrt{(2m-1)^2 + (2n-1)^2}$$

Thus we have

$$u_1(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} (\cosh k_x x + \coth k_x a \sinh k_x x) \cos k_y y \cos k_z z$$

$$u_1(a, x, y) = V_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} (\cosh k_x a + \coth k_x a \sinh k_x a) \cos k_y y \cos k_z z$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 2A_{mn} \cosh k_x a \cos k_y y \cos k_z z$$

Since $\int_{-a}^a \cos \frac{(2m-1)\pi x}{2a} \cos \frac{(2n-1)\pi x}{2a} dx = a \delta_{mn}$ (from the table)

$$A_{mn} = \frac{V_0}{2a^2 \cosh k_x a} \int_{-a}^a dy \int_{-a}^a dz \cos \frac{(2m-1)\pi y}{2a} \cos \frac{(2n-1)\pi z}{2a}$$

$$= \frac{8V_0 (-1)^{m+n}}{(2m-1)(2n-1)\pi^2 \cosh k_x a}$$

(You should be able to do these integrals.)

Now we move to the second problem:

$$\nabla^2 u_2 = 0, \quad u_2(x, \pm a, z) = 0, \quad u_2(x, y, \pm a) = 0, \quad u_2(-a, y, z) = V_0, \quad u_2(a, y, z) = 0$$

This problem is very similar to the previous one.

$$X'' = +k_x^2 X$$

$$X = A \cosh k_x x + B \sinh k_x x$$

$$X(a) = 0 \Rightarrow$$

$$A \cosh k_x a + B \sinh k_x a = 0$$

$$B = -A \coth k_x a$$

$$X(x) \propto \cosh k_x x - \coth k_x a \sinh k_x x$$

$$u_2(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} (\cosh k_x x - \coth k_x a \sinh k_x x) \cos k_y y \cos k_z z$$

$$u_2(-a, 0, 0) = V_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} (\cosh k_x a + \coth k_x a \sinh k_x a) \cos k_y y \cos k_z z$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 2A_{mn} \cosh k_x a \cos k_y y \cos k_z z$$

Note that the A_{mn} are identical to what they were in the previous case.

Adding the two solutions together, we have:

$$u_1(x, y, z) + u_2(x, y, z) = 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \cosh k_x x \cos k_y y \cos k_z z$$

Clearly, we can extend the results to the other two dimensions to arrive at the conclusion:

$$u(x, y, z) = 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} [\cosh k_{mn} x \cos k_m y \cos k_n z + \cos k_m x \cosh k_{mn} y \cos k_n z + \cos k_m x \cos k_n y \cosh k_{mn} z]$$

$$k_m = \frac{(2m-1)\pi}{2a}, \quad k_n = \frac{(2n-1)\pi}{2a}, \quad k_{mn} = \frac{\pi}{2a} \sqrt{(2m-1)^2 + (2n-1)^2}$$

$$A_{mn} = \frac{8V_0(-1)^{m+n}}{(2m-1)(2n-1)\pi^2 \cosh k_{mn} a}$$

Surprisingly enough, this actually sums to V_0 within the cube. Note that the problem is much harder for the outside of the cube.

2. The electrical potential on a sphere of radius a is given by the expression:

$$V(a, \theta, \phi) = \sin(3\phi) \cos^3(2\theta)$$

Find the electric potential for all $r > a$.

Would you expect contributions from Y_0^0 , Y_3^0 , or Y_5^4 in the solution? Why?

(Important ideas: Expanding functions in terms of basis sets, orthonormality of the spherical harmonics with respect to a weighting function, using unstated boundary conditions – finite solution, ϕ dependence of the spherical harmonics.)

You need to know that the solution is of the form:

$$R = A \left(\frac{r}{a} \right)^l + B \left(\frac{a}{r} \right)^{l+1}, \quad l = 0, 1, 2, \dots$$

$$Y_l^m(\theta, \phi)$$

The general solution, then is:

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left[A_{lm} \left(\frac{r}{a} \right)^l + B_{lm} \left(\frac{a}{r} \right)^{l+1} \right] Y_l^m(\theta, \phi)$$

For this to be finite at large values of r , this reduces to:

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l B_{lm} \left(\frac{a}{r} \right)^{l+1} Y_l^m(\theta, \phi)$$

Now we apply the boundary condition above:

$$V(a, \theta, \phi) = \sin(3\phi) \cos^3(2\theta) = \sum_{l=1}^{\infty} \sum_{m=-l}^l B_{lm} Y_l^m(\theta, \phi)$$

$$B_{lm} = \int_0^{2\pi} \int_0^{\pi} \sin(3\phi) \cos^3(2\theta) \bar{Y}_l^m(\theta, \phi) \sin \theta d\theta d\phi$$

You will not need to evaluate the integrals.

The coefficients of the Y_0^0 , Y_3^0 , or Y_5^4 must all be zero, as we know that $\sin(3\phi)$ will only survive in integrals where $m = 3$ or -3 .

3. Solve the following equation:

$$u_{xx} + u_{yy} = 3y\delta\left(x - \frac{a}{2}\right), \quad 0 \leq x \leq a, \quad 0 \leq y \leq a$$

$$u(0, y) = 0, \quad u(a, y) = 0, \quad u(x, 0) = 0, \quad u(x, a) = 0$$

(Important ideas: dealing with non-homogeneous equations by eigenfunction expansion, using the Dirac delta function. The ODE is not so important.)

Homogeneous problem:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$X(x) = \sin \frac{m\pi x}{a}, \quad m = 1, 2, 3, \dots$$

Now we try a solution of the form

$$u(x, y) = \sum_{m=1}^{\infty} A(y) \sin \frac{m\pi x}{a}$$

$$u(x, 0) = \sum_{m=1}^{\infty} A(0) \sin \frac{m\pi x}{a} = 0 \Rightarrow A(0) = 0$$

$$u(x, a) = \sum_{m=1}^{\infty} A(a) \sin \frac{m\pi x}{a} = 0 \Rightarrow A(a) = 0$$

Putting this in the original equation:

$$\sum_{m=1}^{\infty} \left(-\frac{m^2 \pi^2}{a^2} \right) A(y) \sin \frac{m\pi x}{a} + \sum_{m=1}^{\infty} A''(y) \sin \frac{m\pi x}{a} = 3y\delta\left(x - \frac{a}{2}\right)$$

$$\sum_{m=1}^{\infty} \left(A''(y) - \frac{m^2 \pi^2}{a^2} A(y) \right) \sin \frac{m\pi x}{a} = 3y\delta\left(x - \frac{a}{2}\right) = \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{a}$$

$$A''(y) - \frac{m^2 \pi^2}{a^2} A(y) = b_m = \frac{2}{a} \int_0^a 3y\delta\left(x - \frac{a}{2}\right) \sin \frac{m\pi x}{a} dx = \frac{6y}{a} \sin \frac{m\pi}{2}$$

At this point, I would usually give you the solution to this ordinary differential equation, but it is good review to see how we can solve it ourselves.

Take the homogeneous solution to the ODE:

$$A_h(y) = C \sinh \frac{m\pi y}{a} + D \cosh \frac{m\pi y}{a}$$

We add to that a particular solution of the ODE, which can be seen to be:

$$A_p''(y) = 0, \quad -\frac{m^2 \pi^2}{a^2} A_p(y) = \frac{6y}{a} \sin \frac{m\pi}{2}$$

$$A_p(y) = -\frac{6ya}{m^2 \pi^2} \sin \frac{m\pi}{2}$$

Now add the entire particular solution of the PDE to the homogeneous solution of the PDE:

$$A(y) = C \sinh \frac{m\pi y}{a} + D \cosh \frac{m\pi y}{a} - \frac{6ya}{m^2 \pi^2} \sin \frac{m\pi}{2}$$

$$A(0) = 0 = D$$

$$A(a) = C \sinh m\pi - \frac{6a^2}{m^2 \pi^2} \sin \frac{m\pi}{2} = 0$$

$$C = \frac{6a^2}{m^2 \pi^2 \sinh m\pi} \sin \frac{m\pi}{2}$$

$$A(y) = \frac{6a^2}{m^2 \pi^2 \sinh m\pi} \sin \frac{m\pi}{2} \sinh \frac{m\pi y}{a} - \frac{6ya}{m^2 \pi^2} \sin \frac{m\pi}{2}$$

$$u(x, y) = \sum_{m=1}^{\infty} \left[\frac{6a^2}{m^2 \pi^2 \sinh m\pi} \sin \frac{m\pi}{2} \sinh \frac{m\pi y}{a} - \frac{6ya}{m^2 \pi^2} \sin \frac{m\pi}{2} \right] \sin \frac{m\pi x}{a}$$

$$u(x, y) = 6a \sum_{m=1}^{\infty} \frac{1}{m^2 \pi^2} \left[\frac{a}{\sinh m\pi} \sinh \frac{m\pi y}{a} - y \right] \sin \frac{m\pi}{2} \sin \frac{m\pi x}{a}$$

4. Put the following eigenvalue equation in Sturm-Liouville form and write an orthogonality equation for the eigenfunctions. You do not need to solve for the eigenfunctions.

$$y''(x) + \left(\frac{1}{x} + \cot x \right) y'(x) + (\lambda^2 x + \cot x) y(x) = 0, \quad \frac{\pi}{4} \leq x \leq \frac{\pi}{2}$$

$$y\left(\frac{\pi}{4}\right) = 0, \quad y\left(\frac{\pi}{2}\right) - y'\left(\frac{\pi}{2}\right) = 0$$

(Important ideas: Sturm-Liouville form, orthogonality with respect to weighting functions.)

Sturm-Liouville form is:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \quad p(x)y'' + p'(x)y' + [q(x) + \lambda r(x)]y = 0$$

Let's multiply the original equation by $p(x)$:

$$p(x)y''(x) + p(x)\left(\frac{1}{x} + \cot x\right)y'(x) + p(x)(\lambda^2 x + \cot x)y(x) = 0$$

$$p'(x) = p(x)\left(\frac{1}{x} + \cot x\right)$$

$$\int \frac{dp}{p} = \int \left(\frac{1}{x} + \cot x\right) dx$$

$$\ln p = \ln x + \ln(\sin x) + \ln C$$

$$p = Cx \sin x$$

$$p(x) = x \sin x$$

(The integral of $\cot x$ would be given.)

(We can let $C = 1$)

$$x \sin x y''(x) + x \sin x \left(\frac{1}{x} + \cot x \right) y'(x) + x \sin x (\lambda^2 x + \cot x) y(x) = 0$$

$$x \sin x y''(x) + (\sin x + x \cos x) y'(x) + (\lambda^2 x^2 \sin x + x \cos x) y(x) = 0$$

$$\left[x \sin x y'(x) \right]' + (\lambda^2 x^2 \sin x + x \cos x) y(x) = 0$$

$$r(x) = x^2 \sin x$$

$$\int_{\pi/4}^{\pi/2} y_m(x) y_n(x) x^2 \sin x dx \propto \delta_{mn}$$

5. Find the power spectrum of the function $f(t) = H(t-a) - H(t-b)$ where H is the Heaviside function. (The power spectrum is relative power vs. frequency and is given by the absolute square of the Fourier transform.)

(Important ideas: Fourier Transforms and their physical applications, using the Heaviside Function.)

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (H(t-a) - H(t-b)) e^{-i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega t}}{-i\omega} \Big|_a^b = \frac{i}{\sqrt{2\pi}\omega} [e^{-i\omega b} - e^{-i\omega a}] \\ P(\omega) &= |F(\omega)|^2 = \frac{1}{2\pi\omega^2} [e^{-i\omega b} - e^{-i\omega a}] [e^{i\omega b} - e^{i\omega a}] \\ &= \frac{1}{2\pi\omega^2} [2 - e^{-i\omega(b-a)} - e^{+i\omega(b-a)}] \\ &= \frac{1}{\pi\omega^2} [1 - \cos \omega(b-a)] \end{aligned}$$

6. The following is a partial set of Laplace transforms:

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(f'(t)) = sL(f(t)) - f(0)$$

$$L(f''(t)) = s[sL(f(t)) - f(0)] - f'(0)$$

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} L(f(t))(s)$$

$$L(e^{\alpha t} f(t)) = F(s - \alpha) \text{ where } F(s) = L(f(t))(s)$$

$$L(H(t-a)f(t-a))(s) = e^{-as} F(s)$$

$$L(f * g) = L(f)L(g), \quad f * g = \int_0^t f(t-\tau)g(\tau)d\tau$$

$$L(\cos kt) = \frac{s}{s^2 + k^2}, \quad L(\sin kt) = \frac{k}{s^2 + k^2}$$

Solve the equation

$$y'' - y = 1 + t^3, \quad y(1) = 0, \quad y'(1) = 0, \quad t > 1$$

(Important ideas: Using Laplace transforms . Note that the partial fraction expansion would be given.)

$$\text{Let } \tilde{t} = t - 1$$

$$y'' - y = 1 + (\tilde{t} + 1)^3, \quad y(0) = 0, \quad y'(0) = 0, \quad \tilde{t} > 0$$

$$s[sF - y(0)] - y'(0) - F = \frac{2}{s} + \frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2}$$

$$F(s^2 - 1) = \frac{2s^3 + 3s^2 + 6s + 6}{s^4}$$

$$F = \frac{2s^3 + 3s^2 + 6s + 6}{s^4(s^2 - 1)}$$

$$F = -6\frac{1}{s^4} - 6\frac{1}{s^3} - 9\frac{1}{s^2} - 8\frac{1}{s} + \frac{17}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$f(\tilde{t}) = -\tilde{t}^3 - 3\tilde{t}^2 - 9\tilde{t} - 8 + \frac{17}{2}e^{\tilde{t}} - \frac{1}{2}e^{-\tilde{t}}$$

$$f(t) = -(t-1)^3 - 3(t-1)^2 - 9(t-1) - 8 + \frac{17}{2}e^{(t-1)} - \frac{1}{2}e^{-(t-1)}$$

$$= -t^3 - 6t - 1 + \frac{17}{2}e^{(t-1)} - \frac{1}{2}e^{-(t-1)}$$

7. Solve the following problem:

$$u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0$$
$$u(0,t) = e^{-t}, \quad u(x,0) = 0$$

You will need to know

$$L(e^{-\sqrt{s}x}) = \frac{x}{2\sqrt{\pi}} t^{-3/2} e^{-x^2/(4t)}$$

You do not need to evaluate the convolution integral that results.

(Important ideas: Use the Laplace transform to solve PDEs on the semi-infinite range, convolutions.)

$$sU - u(x,0) = U_{xx}$$

$$U_{xx} = sU$$

$$U(x,s) = A(s)e^{-\sqrt{s}x} + B(s)e^{+\sqrt{s}x}$$

$B(s) = 0$ to keep U finite at large x

$$U(0,s) = L(e^{-t}) = \frac{1}{s+1} = A(s)$$

$$U(x,s) = \frac{1}{s+1} e^{-\sqrt{s}x}$$

$$U(x,s) = L(e^{-t})L\left(\frac{x}{2\sqrt{\pi}}t^{-3/2}e^{-x^2/(4t)}\right)$$

$$u(x,t) = (e^{-t}) * \left(\frac{x}{2\sqrt{\pi}}t^{-3/2}e^{-x^2/(4t)}\right)$$

$$= \frac{x}{2\sqrt{\pi}} \int_0^t e^{-(t-\tau)} \tau^{-3/2} e^{-x^2/(4\tau)} d\tau$$

8. Solve the following problem:

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = e^{-x^2/2}, \quad u_t(x,0) = 0$$

You will need the following information.

$$F\left(e^{-x^2/2}\right) = e^{-\omega^2/2}$$

Leave your answer in terms of an inverse Fourier transform integral.

(Important ideas: Using Fourier transforms to solve PDEs.)

$$\hat{u}_{tt} = -\omega^2 \hat{u}, \quad \hat{u}(\omega,0) = e^{-\omega^2/2}, \quad \hat{u}_t(\omega,0) = 0$$

$$\hat{u}(\omega,t) = A \cos \omega t + B \sin \omega t$$

$$\hat{u}_t(\omega,t) = -A \omega \sin \omega t + B \omega \cos \omega t$$

$$\hat{u}_t(\omega,0) = B \omega = 0$$

$$\hat{u}(\omega,0) = A = e^{-\omega^2/2}$$

$$\hat{u}(\omega,t) = e^{-\omega^2/2} \cos \omega t$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\omega^2/2} \cos \omega t e^{i\omega x} d\omega$$