Physics 318
Orthogonal and Orthonormal Functions

Orthogonal functions are functions that satisfy a relationship such as:

$$\int_a^b \varphi_{nk}(q)\varphi_{n\ell}(q)w(q)dq = A_{nk}\delta_{kl}$$

• The functions are defined over the range [a,b].
• The indices may indicate different functions or different constants in the arguments. For example, we may write:

$$\varphi_n(x) \propto \sin\frac{n\pi x}{L}$$

$$\varphi_{nm}(x) \propto J_n\left(\frac{\alpha_{nm}r}{a}\right)$$

• $w(q)$ is a weighting function
• $\delta_{mn} = \begin{cases} 1, & if \ \ m = n \\ 0, & if \ \ m \neq n \end{cases}$

Orthonormal functions are similar, except they satisfy the relationship:

$$\int_a^b \varphi_{nk}(q)\varphi_{n\ell}(q)w(q)dq = \delta_{kl}$$

A set of orthogonal functions is a basis set if all piecewise smooth functions can be expanded in terms of the set of functions. We sometimes express this property of a basis set by saying that the basis set “spans the space.” A basis set composed of orthonormal functions is said to be an “orthonormal basis set.” The set of solutions of a differential equation often is an orthonormal basis set. (Sturm-Liouville Theory, Chapter 6.)

The following pages list some useful sets of functions.

1. Sines and cosines I.	
functions	$X(x) = A\cos\mu_n x + B\sin\mu_n x$
diff. eq.	$X''(x) = -\mu_n X(x)$
Range	*0* to *L*
μ_n	$\mu_n = \frac{n\pi}{L}, \quad \frac{(n-1/2)\pi}{L}$, etc., depending on boundary conditions mixed b.c. lead to transcendental equations
orthogonality	$\int_0^L \sin\mu_n x \sin\mu_m x dx = \frac{L}{2}\delta_{mn}, \quad \int_0^L \cos\mu_n x \cos\mu_m x dx = \frac{L}{2}\delta_{mn}, \quad \int_0^L \sin\mu_n x \cos\mu_m x dx = 0$ This holds for both μ_n listed above, but solutions to transcendental equations have a different normalization factor.
orthonormal functions	$\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\mu_n x, \quad \sqrt{\frac{2}{L}}\cos\mu_n x$, weighting function = 1
used with	wave, heat transfer, and Laplace's equations in Cartesian coordinates

2. Sines and cosines II.	
functions	$X(x) = A\cos\mu_n x + B\sin\mu_n x$
diff. eq.	$X''(x) = -\mu_n X(x)$
Range	–*L* to *L*
μ_n	$\mu_n = \frac{n\pi}{L}, \quad \frac{(n-1/2)\pi}{L}$, etc., depending on boundary conditions mixed b.c. lead to transcendental equations
orthogonality	$\int_0^L \sin\mu_n x \sin\mu_m x dx = L\delta_{mn}, \quad \int_0^L \cos\mu_n x \cos\mu_m x dx = L\delta_{mn}, \quad \int_0^L \sin\mu_n x \cos\mu_m x dx = 0$ This holds for both μ_n listed above, but solutions to transcendental equations have a different normalization factor.
orthonormal functions	$\varphi_n(x) = \sqrt{\frac{1}{L}}\sin\mu_n x, \quad \sqrt{\frac{1}{L}}\cos\mu_n x$, weighting function = 1
used with	wave, heat transfer, and Laplace's equations in Cartesian coordinates

3. Sines and cosines III.	
functions	$\Theta(\theta) = A\cos m\theta + B\sin m\theta$
diff. eq.	$\Theta''(x) = -\mu_n \Theta(\theta)$
Range	0 to 2π
orthogonality	$\int_0^{2\pi} \sin mx \sin nx dx = \pi\delta_{mn}$, $\int_0^{2\pi} \cos mx \cos nx dx = \pi\delta_{mn}$, $\int_0^{2\pi} \sin mx \cos nx dx = 0$
orthonormal functions	$\varphi_m(\theta) = \sqrt{\frac{1}{\pi}} \sin m\theta, \quad \sqrt{\frac{1}{\pi}} \cos m\theta$, weighting function = 1
used with	θ dependence in polar and cylindrical coordinates ϕ dependence in spherical coordinates – when the angular part of the equation is the same as for Laplace's equation

4. Hyperbolic sines and cosines	
functions	$X(x) = A\cosh \mu_n x + B\sinh \mu_n x$
diff. eq.	$X''(x) = +\mu_n X(x)$
μ_n	$\mu_n = \frac{n\pi}{L}, \quad \frac{(n-1/2)\pi}{L}$, depending on boundary conditions
orthogonality	Not an orthogonal set of functions
used with	Laplace's equation in Cartesian coordinates in more than one dimension Solutions will be sin and cos in one dimension and sinh and cosh in another.
useful info	$\frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x$

5. Bessel functions	
functions	$R(r) = AJ_m(\lambda_{mn} r) + BY_m(\lambda_{mn} r)$
diff. eq.	$r^2 R'' + rR' + (\lambda_{mn}^2 r^2 - m^2)R = 0$
λ_{mn}	$\lambda_{mn} = \dfrac{\alpha_{mn}}{a}, \quad u(a,\theta) = 0, \quad \alpha_{mn}$ are the zeros of the Bessel function
range	*0* to *a*
orthogonality	$\int_0^a J_m(\lambda_{mn} r) J_m(\lambda_{mn'} r) r dr = \dfrac{a^2}{2} J_{m+1}^2(\alpha_{mn}) \delta_{nn'}$
orthonormal functions	$\varphi_{mn}(r) = \dfrac{\sqrt{2}}{aJ_{m+1}(\alpha_{mn})} J_m(\lambda_{mn} r)$, weighting function = *r*
used with	wave equation and heat transfer equations in polar coordinates and equations in cylindrical coordinates
useful info	$Y_m(0)$ is not finite
asymptotic forms	For large *r*: $J_m(kr) \to \sqrt{\dfrac{2}{\pi k r}} \cos\left[kr - \dfrac{\pi}{4}(1+2m)\right], \quad Y_m(kr) \to \sqrt{\dfrac{2}{\pi k r}} \sin\left[kr - \dfrac{\pi}{4}(1+2m)\right]$

6. Euler's equation I.	
functions	$R(r) = A + b\ln r,\ m = 0; \quad R(r) = A\left(\dfrac{r}{a}\right)^m + B\left(\dfrac{a}{r}\right)^m,\ m > 0$
diff. eq.	$r^2 R'' + rR' - m^2 R = 0$
Range	*0* to ∞
orthogonality	Not an orthogonal set of functions
used with	Laplace's equation in polar coordinates

7. Euler's equation II.	
functions	$R(r) = A\left(\frac{r}{a}\right)^{\ell} + B\left(\frac{a}{r}\right)^{\ell+1}$
diff. eq.	$r^2 R'' + 2rR' - \ell(\ell+1)R = 0$
Range	0 to ∞
orthogonality	Not an orthogonal set of functions
used with	Laplace's equation in spherical coordinates

8. Associated Legendre polynomials	(Or if $\ell = 0$, regular Legendre polynomials)
functions	$P_\ell^m(\cos\theta)$
diff. eq.	$\Theta'' + \cot\theta\Theta' + \left[\ell(\ell+1) - m^2\csc^2\theta\right]\Theta = 0$
Range	0 to π
orthogonality	$\int_0^\pi P_\ell^m(\cos\theta)P_{\ell'}^m(\cos\theta)\sin\theta d\theta = \delta_{\ell\ell'}\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}$ weighting function=sin(θ)
orthonormal functions	See spherical harmonics
used with	θ dependence in polar and spherical coordinates when the angular part of the equation is the same as for Laplace's equation
useful info	$P_\ell^m(\theta)$ has terms in $\cos^{\ell-\lvert m\rvert}\theta\sin^{\lvert m\rvert}\theta,\quad \cos^{\ell-\lvert m\rvert-2}\theta\sin^{\lvert m\rvert}\theta,\quad \ldots\quad \ell-\lvert m\rvert-2n \geq 0$ Hence, the only non-zero terms have $m = -\ell, -\ell+1, \ldots, +\ell-1, +\ell$

9. Spherical harmonics	
functions	$Y_\ell^m(\theta,\phi)$
diff. eq.	$Y_{\theta\theta} + \cot\theta\, Y_\theta + \csc^2\theta\, Y_{\phi\phi} + \ell(\ell+1)Y = 0$
Range	$0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi$
orthogonality	$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi)\overline{Y_{\ell'}^{m'}(\theta,\phi)}\sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\delta_{mm'}$ weighting function=sin(θ) Note that we take the complex conjugate of the second spherical harmonic.
used with	θ and ϕ dependence in spherical coordinates when the angular part of the equation is the same as for Laplace's equation
useful info	$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}$

10. Spherical Bessel and Neumann functions	
functions	$R(r) = Aj_\ell(kr) + Bn_\ell(kr)$
diff. eq.	$r^2R'' + 2rR' + \left(k^2r^2 - \ell(\ell+1)\right)R = 0$
range	*0* to *a* (or *0* to ∞, depending on the problem)
orthogonality	$\int_0^a j_\ell\left(\frac{\alpha_{\ell n}r}{a}\right) j_\ell\left(\frac{\alpha_{\ell n'}r}{a}\right) r^2 dr = \frac{a^3}{2}\left[j_{\ell+1}(\alpha_{\ell n})\right]^2\delta_{nn'}$ useful when b.c. are $R(a) = 0$.
used with	wave, heat transfer, and free-particle Schrödinger equations ($V(r) = 0$) in spherical coordinates
useful info	$n_\ell(0)$ is not finite $j_\ell(kr) = \sqrt{\frac{\pi}{2kr}}J_{\ell+1/2}(kr), \quad n_\ell(kr) = (-1)^{\ell+1}\sqrt{\frac{\pi}{2kr}}J_{-\ell-1/2}(kr)$
asymptotic forms	For large *r*: $j_\ell(kr) \to \frac{1}{kr}\cos\left[kr - \frac{(\ell+1)\pi}{2}\right], \quad n_\ell(kr) \to \frac{1}{kr}\sin\left[kr - \frac{(\ell+1)\pi}{2}\right]$