Exercise 3.9.

2 and 4. See Maple worksheets.

4. In addition to the Maple worksheet, we need to work out the details of the solution to Laplace’s Equation with the one non-homogeneous boundary condition specified.

\[ u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 1 \]
\[ u(x,0) = u(0, y) = u(1, y) = 0 \]
\[ u(x,1) = x \]
\[ X(0) = X(1) = Y(0) = 0, \quad Y(1) = x \]
\[ -\frac{Y''}{Y} = \frac{X''}{X} = -\mu^2 \]
\[ X(x) = \sin \mu_n x, \quad \mu_n = n\pi \]
\[ Y(y) = A \cosh \mu_n y + B \sinh \mu_n y \]
\[ Y(0) = A = 0 \]
\[ u(x, y) = \sum_{n=1}^{\infty} a_n \sin \mu_n x \sinh \mu_n y \]
\[ u(x,1) = x = \sum_{n=1}^{\infty} a_n \sin \mu_n x \sin \mu_n x = \sum_{n=1}^{\infty} b_n \sin \mu_n x \]
\[ b_n = 2 \int_0^1 x \sin \mu_n x \]
\[ a_n = \frac{2 \int_0^1 x \sin \mu_n x}{\sinh \mu_n} \]
7. Also refer to the Maple worksheet.

\[ u_{xx} + u_{yy} = \sin 2\pi x, \quad 0 < x < 1, \quad 0 < y < 1 \]
\[ u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0 \]
\[ X(x) = \sin(n\pi x) \]

\[ u(x, y) = \sum_{n=1}^{\infty} a_n(y) \sin n\pi x \]

\[ u(x, 0) = \sum_{n=1}^{\infty} a_n(0) \sin n\pi x = 0 \Rightarrow a_n(0) = 0 \]

\[ u(x, 1) = \sum_{n=1}^{\infty} a_n(1) \sin n\pi x = 0 \Rightarrow a_n(1) = 0 \]

Now put \( u(x, y) \) in the differential equation.

\[ -\sum_{n=1}^{\infty} a_n(y)n^2\pi^2 \sin n\pi x + \sum_{n=1}^{\infty} a''_n(y) \sin n\pi x = \sin 2\pi x \]

\[ \sum_{n=1}^{\infty} [a''_n(y) - n^2\pi^2 a_n(y)] \sin n\pi x = \sin 2\pi x \]

By inspection, only one term survives in the sum:

\[ a''_2(y) - 4\pi^2 a_2(y) = 1, \quad a_2(0) = a_2(1) = 0 \]

\[ a_2(y) = \frac{e^{2\pi y} + e^{2\pi(1-y)} - (1 + e^{2\pi})}{4\pi^2 (1 + e^{2\pi})} \text{ by Maple} \]

\[ a''_n(y) - n^2\pi^2 a_n(y) = 0, \quad a_n(0) = a_n(1) = 0, \quad n \neq 2 \]

\[ a_n(y) = 0 \text{ by Maple or simple math} \]

\[ u(x, y) = \frac{e^{2\pi y} + e^{2\pi(1-y)} - (1 + e^{2\pi})}{4\pi^2 (1 + e^{2\pi})} \sin 2\pi x \]
The problem does not require it, but the solution can be simplified (Maple didn’t work well to simplify it, however!) and the solution checked.

\[
\begin{align*}
    u(x, y) &= \frac{e^{2\pi y} + e^{2\pi(1-y)} - (1 + e^{2\pi})}{4\pi^2 (1 + e^{2\pi})} \sin 2\pi x \\
    &= \frac{e^{\pi(2y-1)} + e^{-\pi(2y-1)} - (e^{-\pi} + e^{\pi})}{4\pi^2 (e^{-\pi} + e^{\pi})} \frac{e^{\pi}}{e^{\pi}} \sin 2\pi x \\
    &= \frac{\cosh[\pi(2y-1)] - \cosh \pi}{4\pi^2 \cosh \pi} \sin 2\pi x \\
    u_{xx} + u_{yy} &= -4\pi^2 \frac{\cosh[\pi(2y-1)] - \cosh \pi}{4\pi^2 \cosh \pi} \sin 2\pi x \\
    &= \frac{4\pi^2 \cosh \pi}{4\pi^2 \cosh \pi} \sin 2\pi x = \sin 2\pi x
\end{align*}
\]

Note that this form of the solution is much simpler than the double summation form.