Exercise 3.7.

A. See the maple worksheet.

B. 
\[ u_t = c^2 (u_{xx} + u_{yy}), \quad 0 < x < a, \quad 0 < y < b \]
\[ u(x,0,t) = 0, \quad u(0,y,t) = 0 \]
\[ u(x,b,t) = 0, \quad u(a,y,t) = 0 \]
\[ u(x,y,0) = f(x) \]

I. Separation of variables, boundary conditions

\[ u(x,y,t) = X(x)Y(y)T(t) \]
\[ \frac{T'}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = -k^2, \quad X(0) = X(a) = 0, \quad Y(0) = Y(b) = 0 \]
\[ \frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2 = -\mu^2 \]
\[ \frac{Y''}{Y} = -(k^2 - \mu^2) = -\nu^2, \quad k^2 = \mu^2 + \nu^2 \]

From our previous experience in one dimension with similar boundary conditions, we conclude that the separation variables must be negative. (Be sure you know how to prove that the other cases lead to trivial solutions.)

II. Solve for X, Y, and T

\[ X'' = -\mu^2 X, \quad Y'' = -\nu^2 Y, \]
\[ X = A \cos \mu x + B \sin \mu x \quad Y = A \cos \nu y + B \sin \nu y \]
\[ X(0) = 0 \Rightarrow A = 0 \quad Y(0) = 0 \Rightarrow A = 0 \]
\[ X(a) = 0 \Rightarrow \mu_m a = m\pi \quad Y(b) = 0 \Rightarrow \nu_n b = n\pi \]
\[ X(x) = \sin \mu_m x \quad Y(y) = \sin \nu_n y \]

\[ T' = -c^2 k^2 T = -\lambda_{mn}^2 T, \]
\[ \lambda_{mn} = ck = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \]
\[ T(t) = e^{-\lambda_{mn} t} \]
III. Make a series solution

\[ u(x, y, t) = \sum_{n=1}^{\infty} A_{mn} \sin \mu_n x \sin \nu_n y \ e^{-\frac{\lambda_n^2}{4}t} \]

\[ u(x, y, 0) = f(x, y) = \sum_{n=1}^{\infty} A_{mn} \sin \mu_n x \sin \nu_n y \]

We can solve for the coefficients by using the orthogonality conditions on the sine functions:

\[ A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \mu_n x \sin \nu_n y \ dy \ dx \]