Exercise 2.6

1 and 3. See the Maple worksheets.

9, 12, 13

9. We may assume the orthogonality relations for sine and cosine on p.22 hold.

\[
\frac{1}{2p} \int_{-p}^{p} e^{\frac{imx}{p}} e^{\frac{-in\pi x}{p}} \, dx = \frac{1}{2\pi} \int_{-n}^{n} e^{imu} e^{-inx} \, du \\
= \frac{1}{2\pi} \int_{-n}^{n} [\cos mu + i\sin mu][\cos nu - i\sin nu] \, du \\
= \frac{1}{2\pi} \int_{-n}^{n} [\cos mu \cos nu + \sin mu \sin nu - i\cos mu \sin nu + i\sin mu \cos nu] \, du \\
= \frac{1}{2\pi} \int_{-n}^{n} [\cos(mu - nu)] \, du \\
= \delta_{mn}
\]

12. We take the result of Example 1:

\[
e^{ax} = \frac{\sinh {\pi a}}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2} (a + in) e^{inx} \\
= \frac{\sinh {\pi a}}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2} (a + in)(\cos nx + i \sin nx) \\
= \frac{\sinh {\pi a}}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2} [(a \cos nx - n \sin nx) + i(n \cos nx + a \sin nx)]
\]

Since \( a \) is real, \( e^{ax} \) is also real, so the imaginary part of the sum must be zero, and:

\[
e^{ax} = \frac{\sinh {\pi a}}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2} [a \cos nx - n \sin nx]
\]
13. We can use the result of Problem 12 with \( x = \pi \). Since the expansion is from -\( \pi \) to +\( \pi \), the Fourier series approaches \( e^{a\pi} \) from the left and \( e^{-a\pi} \) from the right. (The Fourier series is 2\( \pi \) periodic, and starts repeating itself to right of \( x = \pi \).) Just at \( x = \pi \), the series sums to the average of these two values:

Fourier sum at \( x = \pi \) is: \[ \frac{e^{a\pi} + e^{-a\pi}}{2} = \cosh a\pi \]. (Compare Examples 1 and 2.)

\[
\cosh a\pi = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2} [a \cos n\pi - n \sin n\pi]
\]

\[
\cosh a\pi = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2} a(-1)^n
\]

\[
\coth a\pi = \frac{a}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2}
\]

\[
\coth t = \frac{t}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(t/\pi)^2 + n^2}
\]

\[
\coth t = \sum_{n=-\infty}^{\infty} \frac{t}{t^2 + \pi^2 n^2}
\]