

The Associated Legendre Equation

$$(1 - x^2)y'' - 2xy' + \left(\mu - \frac{m^2}{1 - x^2}\right)y = 0$$

$$(1 - 2x^2 + x^4)y'' - 2x(1 - x^2)y' + [\mu(1 - x^2) - m^2]y = 0$$

$$y'' - 2x^2y'' + x^4y'' - 2xy' + 2x^3y' + (\mu - m^2)y - \mu x^2y = 0$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$$

$$y'' = 2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + \dots$$

x^0	x^1	x^2	x^3	x^4	x^n
$2a_2$	$6a_3$	$12a_4$	$20a_5$	$30a_6$	$(n+2)(n+1)a_{n+2}$
		$-4a_2$	$-12a_3$	$-24a_4$	$-2n(n-1)a_n$
				$2a_2$	$(n-2)(n-3)a_{n-2}$
	$-2a_1$	$-4a_2$	$-6a_3$	$-8a_4$	$-2na_n$
			$2a_1$	$4a_2$	$2(n-2)a_{n-2}$
$(\mu - m^2)a_0$	$(\mu - m^2)a_1$	$(\mu - m^2)a_2$	$(\mu - m^2)a_3$	$(\mu - m^2)a_4$	$(\mu - m^2)a_n$
		$-\mu a_0$	$-\mu a_1$	$-\mu a_2$	$-\mu a_{n-2}$

For even values of n :

$$a_2 = \frac{m^2 - \mu}{2} a_0$$

$$a_4 = \frac{8 + m^2 - \mu}{12} a_2 + \frac{\mu}{12} a_0$$

For odd values of n :

$$a_3 = \frac{2 + m^2 - \mu}{6} a_1$$

$$a_5 = \frac{18 + m^2 - \mu}{20} a_3 + \frac{\mu - 2}{20} a_1$$

And in general, for $n > 1$:

$$a_{n+2} = \frac{2n^2 + m^2 - \mu}{(n+2)(n+1)} a_n + \frac{\mu - (n-1)(n-2)}{(n+2)(n+1)} a_{n-2}$$

If we put in arbitrary values of μ and n , then the coefficients can be evaluated. For example, with $m = 0$ and $\mu = 33$, we have:

$$a_{10} = -2.455781250$$

$$a_{20} = -3.629891661$$

$$a_{30} = -5.035530493$$

$$a_{40} = -6.450607143$$

$$a_{50} = -7.867259031$$

We see that these don't get small as n gets large. In fact, the series does not converge. As we see from the recursion relation, we could force the series to converge if let some consecutive coefficients be zero. But is that possible for all the coefficients above a certain value of n to be zero? Let's say, for example, that the largest n is 8. Then:

$$0 = \frac{\mu - (10-1)(10-2)}{(n+2)(n+1)} a_{10-2}$$

$$\mu = 72 = n_{\max}(n_{\max} + 1)$$

with $m = 0$ and $\mu = 72$, we have:

$$a_0 = 1$$

$$a_2 = -36$$

$$a_4 = 198$$

$$a_6 = -343.2$$

$$a_8 = 183.8571429$$

$$a_{10} = 0$$

$$a_{12} = 0$$

We just assumed we could write $m = 0$, but what about other values of m ? We know that m must be an integer from separation of variables, but otherwise there are no *a priori* restrictions on it. But we have one piece of information we didn't use above: we know a_{10} is also zero, so:

$$a_{10} = 0 = \frac{128 + m^2 - 72}{90} a_8 + \frac{72 - 42}{90} a_6$$

If we let $a_1 = 1$ and continue the recursion relations on down to, we can show that, for arbitrary m :

$$a_{10} = 0 = \frac{1}{3628800} m^{10} - \frac{1}{30240} m^8 + \frac{13}{10800} m^6 - \frac{41}{2835} m^4 + \frac{64}{1575} m^2$$

This leads to the result that

$$m = 0, \pm 2, \pm 4, \pm 6, \pm 8$$

In other words, these series will only terminate for even values of m ranging from $-n_{max}$ to $+n_{max}$. What about other integral values of m ? We can get an idea of what will happen by looking at the coefficients for a few other cases, still with $\mu = 72$, $n_{max} = 8$.

$m = 10$	$m = 3$	$m = 9$
$a_0 = 1$	$a_0 = 1$	$a_0 = 1$
$a_2 = 14$	$a_2 = -315$	$a_2 = 4.5$
$a_4 = 48$	$a_4 = 150.375$	$a_4 = 12.375$
$a_6 = 126.8$	$a_6 = -224.6875$	$a_6 = 26.8125$
$a_8 = 271$	$a_8 = 103.523438$	$a_8 = 50.273438$
$a_{10} = 512$	$a_{10} = -0.128906$	$a_{10} = 85.464844$
$a_{12} = 884.36363$	$a_{12} = -0.133789$	$a_{12} = 135.319336$
$a_{14} = 1428.58741$		$a_{50} = 9031.039495$

We can guess (and the guess is correct) that if $m > n_{max}$, the series does not converge. However, for $m = 3$, the series seems to converge without the coefficients going to zero. Further analysis shows that for odd values of m , the result is a polynomial of a finite number of terms multiplied by $\sqrt{1 - x^2}$.

We didn't say anything about the coefficients of odd order. These are useful when n_{max} is an odd integer. The same basic rules apply.

We can easily summarize these results by letting $x = \cos \theta$. Then the solutions require $\mu = \ell(\ell + 1)$, $\ell = 0, 1, 2, \dots$ and $m = -\ell, -\ell + 1, \dots, +\ell$. The solutions are of the general form:

$$P_\ell^m(\theta) = \sin^{|m|} \theta \left(C_0 \cos^{\ell-|m|} \theta + C_2 \cos^{\ell-|m|-2} \theta + \dots + C_k \cos^k \theta \right) \text{ where } k \text{ is either } 0 \text{ or } 1.$$

