Appendix B – The Relativistic Transformation of Forces

B–1. The Four-force

We introduced the idea of forces in Chapter 3 where we saw that the change in the four-momentum per unit time is given by the expression

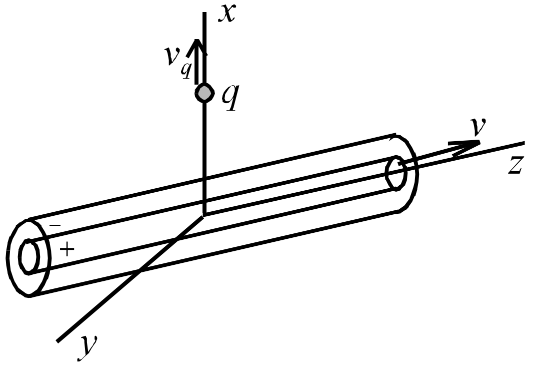
$$\frac{d\boldsymbol{E}}{d\,w} = \begin{pmatrix} \vec{\beta}\cdot\boldsymbol{F} \\ F_x \\ F_y \\ F_z \end{pmatrix} .$$

Remember that the four-vector $\boldsymbol{E}$ has spatial components that are $p_x c$, etc. so we take the derivative with respect to $w=ct$ to get the components of the force.

For a quantity to be a true four-vector, it must transform according to the Lorentz Transformation. Since the four-momentum is a true four-vector, $\Delta\boldsymbol{E}$ must also be a four-vector. But when we divide this by Δt, we no longer have a four-vector, because of the way time transforms. If we could divide $\Delta\boldsymbol{E}$ by a quantity that would be the same in all reference frames, we would have a four-vector, however. To this end, we define “proper time” as time measured in the rest frame of an object. (Note that “proper” means “one’s own” in this context.) If we use proper time in every frame, then all measurements of time must agree. To define the four-force as a true four-vector, all we need to do is know the relationship of proper time to the time measured in a reference frame **S**. This is given in Section 5–3 as $\Delta t = \gamma\Delta\tau$ where τ represents the proper time and γ is a function of the velocity v of the particle as measured in the frame **S**. Then:

$$\boldsymbol{f} = \frac{d\boldsymbol{E}}{d\,c\tau} = \gamma\frac{d\boldsymbol{E}}{d\,ct} = \gamma\begin{pmatrix} \vec{\beta}\cdot\boldsymbol{F} \\ F_x \\ F_y \\ F_z \end{pmatrix} . \qquad \text{(B–1)}$$

B–2. Moving Rods.



Let us take two infinitely-long, coaxial, cylindrical rods. The inner rod is positively charged and the outer, hollow rod is negatively charged. In a reference frame **S**, the outer rod is stationary and the inner rod moves along the $+z$ axis with a velocity v. A point charge q is located a distance r from the rod along the $+x$ axis. It is moving with a velocity v_q in the $+x$ direction. The rods have linear charge densities (charge per unit length) of $\pm\lambda_0$ as measured in **S**. Find the force on the charge given that the force on q from a positive rod at rest is

$$\boldsymbol{F} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{x},$$

where λ is the linear charge density of the rod and ε_0 is a constant called the “permittivity of free space.”

We know that the force from the negative rod is just $\boldsymbol{F}_- = -\frac{\lambda}{2\pi\varepsilon_0 r}\hat{x}$, But to find the force from the moving, positive rod, we must first boost the charge into the rest frame of the positive rod. Let us call the rest frame of the positive rod **S′** and denote quantities in this frame by primes. To transform to this frame, we need a boost of velocity *v* along the +*z* direction:

$$\boldsymbol{L} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

We can summarize the four-vectors as follows:

	space-time	energy-momentum
In **S**	$\boldsymbol{x} = \begin{pmatrix} 0 \\ r \\ 0 \\ 0 \end{pmatrix}$	$\boldsymbol{E} = \begin{pmatrix} E \\ pc \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_0 E_0 \\ \beta_0\gamma_0 E_0 \\ 0 \\ 0 \end{pmatrix}$
In **S′**	$\boldsymbol{x}' = \boldsymbol{L}\,\boldsymbol{x} = \begin{pmatrix} 0 \\ r \\ 0 \\ 0 \end{pmatrix}$	$\boldsymbol{E}' = \boldsymbol{L}\boldsymbol{E} = \begin{pmatrix} \gamma\gamma_0 E_0 \\ \beta_0\gamma_0 E_0 \\ 0 \\ -\beta\gamma\gamma_0 E_0 \end{pmatrix} = \begin{pmatrix} E' \\ p'_x c \\ 0 \\ p'_z c \end{pmatrix}$

In this table, γ is a function of v and γ_0 is a function of v_0.

In **S′** the force on *q* is then:

$$\boldsymbol{F}' = \frac{\lambda_+}{2\pi\varepsilon_0}\frac{q}{r}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Note that the linear charge density is different than in **S** because of the contraction of the rod in the **S** frame. Now, we wish to make a four-force out of this by using Equation (B–1) so we can transform the force back into **S**. To do this, we first find the time component of the four-force:

$$\vec{\beta}'\cdot\boldsymbol{F}' = F'\beta'_x = \frac{\lambda_+}{2\pi\varepsilon_0}\frac{q}{r}\frac{p'_x c}{E'} = \frac{\lambda_+}{2\pi\varepsilon_0}\frac{q}{r}\frac{\beta_0}{\gamma}$$

The four-force is then:

$$\boldsymbol{f}' = \gamma'\begin{pmatrix} \vec{\beta}\cdot\boldsymbol{F}' \\ F'_x \\ F'_y \\ F'_z \end{pmatrix} = \frac{\lambda_+}{2\pi\varepsilon_0}\frac{q}{r}\gamma'\begin{pmatrix} \frac{\beta_0}{\gamma} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We now boost this back to the **S** frame using the inverse transformation.

$$\boldsymbol{F} = \boldsymbol{L}^{-1}\boldsymbol{F}' = \frac{\lambda_+}{2\pi\varepsilon_0}\frac{q}{r}\gamma' \begin{pmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{\beta_0}{\gamma} \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\lambda_+}{2\pi\varepsilon_0}\frac{q}{r}\gamma' \begin{pmatrix} \beta_0 \\ 1 \\ 0 \\ \beta\beta_0 \end{pmatrix}$$

From this we can extract the three-force.

$$\boldsymbol{F}^3 = \frac{\lambda_+}{2\pi\varepsilon_0}\frac{q}{r}\frac{\gamma'}{\gamma_0} \begin{pmatrix} 1 \\ 0 \\ \beta\beta_0 \end{pmatrix}$$

We now need to find γ' and λ_+ in terms of quantities defined in **S**:

$$\gamma' = \frac{E'}{E_0} = \frac{\gamma\gamma_0 E_0}{E_0} = \gamma\gamma_0$$

And, since in **S**, the positive rod is contracted by a factor of γ, a function of v, the charge must appear to be larger in **S** by the same factor:

$$\lambda_0 = \lambda_+\gamma .$$

Thus:

$$\boldsymbol{F}^3{}_+ = \frac{\lambda_+}{2\pi\varepsilon_0}\frac{q}{r}\frac{\gamma\gamma_0}{\gamma_0} \begin{pmatrix} 1 \\ 0 \\ \beta\beta_0 \end{pmatrix} = \frac{\lambda_0}{2\pi\varepsilon_0}\frac{q}{r} \begin{pmatrix} 1 \\ 0 \\ \beta\beta_0 \end{pmatrix}$$

Finally, we add this to the force on q from the negative rod to get the total force on q:

$$\boldsymbol{F}^3 = -\frac{\lambda_0}{2\pi\varepsilon_0}\frac{q}{r} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\lambda_0}{2\pi\varepsilon_0}\frac{q}{r} \begin{pmatrix} 1 \\ 0 \\ \beta\beta_0 \end{pmatrix} = \frac{\lambda_0}{2\pi\varepsilon_0}\frac{q}{r} \begin{pmatrix} 0 \\ 0 \\ \beta\beta_0 \end{pmatrix} \qquad \text{(B–2)}$$

We see that electrostatic forces of the positive and negative charges cancel out to give a residual force in the z direction. The origin of a force in the z direction is seen to be the Lorentz transformation that mixes the t component with the z component of the four-force. The magnitude of this force is smaller than the electrostatic force of a single rod by a factor of $\beta\beta_0$, which is a very small quantity if the velocities are small.

With a little algebra and using the identity $c^2 = 1 / \mu_0\varepsilon_0$ where μ_0 is another constant, called the "permeability of free space," we can put this force in an interesting form:

$$\boldsymbol{F}^3 = \frac{\lambda_0}{2\pi\varepsilon_0}\frac{q}{r}\beta\beta_0\hat{z} = \frac{\lambda_0}{2\pi\varepsilon_0}\frac{q}{r}\frac{vv_0}{c^2}\hat{z} = \frac{\lambda_0\varepsilon_0\mu_0}{2\pi\varepsilon_0}\frac{q}{r}\frac{dz}{dt}v_0\hat{z} = qv_0\frac{\mu_0}{2\pi r}\frac{dq}{dt}\hat{z} = qv_0\frac{\mu_0 i}{2\pi r}\hat{z} = qv_0 B(r)\hat{z}$$

where i is the current (charge per unit time) carried by the positive rod, and $B(r)$ is the magnetic field of a long wire carrying current *i*. In other words, the residual, velocity-dependent force resulting from the Lorentz transformation, is what is usually thought of as the force resulting from a magnetic field produced by a moving charge.

These results can be generalized to obtain the force on a charge *q* moving with velocity $\boldsymbol{v}^{3}{}_{q}$ from a rod of charge density λ moving with velocity $\boldsymbol{v}^{3}{}_{r}$. The charge is a distance *r* from the rod.

$$\boldsymbol{F_0} = \frac{\lambda}{2\pi\varepsilon_0}\frac{q}{r}\hat{r}$$

$$\boldsymbol{F} = \boldsymbol{F_0} + \frac{1}{c^2}\boldsymbol{v_q}\times(\boldsymbol{v_r}\times\boldsymbol{F_0})$$

The part of the force that has no velocity dependence is thought of as resulting from an electric field, ***E***, and the part with velocity dependence is thought of as resulting from a magnetic field, ***B***. In terms of fields, we have:

$$\boldsymbol{F} = q\boldsymbol{E} + q\,\boldsymbol{v_q}\times\boldsymbol{B}$$

$$\boldsymbol{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

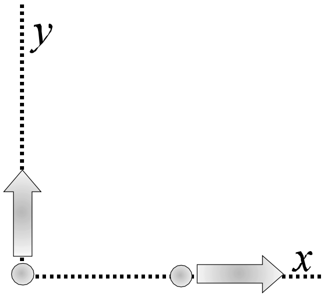
$$\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{v_r}\times\boldsymbol{E}$$

In fact, we can derive all of Maxwell’s Equations by starting from Coulomb’s Law for source charges at rest and applying the Lorentz transformation to forces on test charges. With this and with the assumption that electric and magnetic fields propagate at the speed of light, all of classical (not quantum) electrodynamics can be generated. Another way of saying the same thing is that special relativity is built into classical electrodynamics. It was the apparent conflict of Maxwellian electrodynamics and Newtonian mechanics that led Lorentz and Einstein to the special theory of relativity in the first place.

B–3. Gravitational Magnetism and Newton’s Third Law

Newton’s Law of Gravity and Coulomb’s Law of Electrostatics are very similar inverse square force laws. It is clear that if electrostatic forces have associated with them velocity-dependent forces because of relativistic effects, then gravitational forces must also have similar components to their forces. We may consider these velocity dependent gravitational forces to be caused by “gravitational magnetism.” With electrostatic forces, however, there can be attraction and repulsion that cancel each other out, leaving only the magnetic force as a residual force. In the gravitational case, the effects of gravitational magnetism must be tiny compared to the gravitational force itself, particularly if bodies are not moving near the speed of light. Let us look at the effects of gravitational magnetism in a simple system.

There are two particles in a reference frame **S**. Particle 1 is located at the origin and is moving in the +*y* direction. Particle 2 is located along the *x* axis and is moving in the +*x* direction. Find the forces on the particles assuming that the correct form for the gravitational three-force on particle 2 when particle 1 is at rest is:



$$\boldsymbol{F_{21}} = \frac{\tilde{G}E_{10}E_2}{r_{21}^2}\hat{r}_{21}$$

where E_{10} is the rest energy of particle 1,
E_2 is the total energy of particle 2,
and $\tilde{G} = \frac{G}{c^4}$.

In the **S** frame, the four-positions and four-momenta are:

$$\boldsymbol{x_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{x_2} = \begin{pmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{E_1} = \begin{pmatrix} E_1 \\ 0 \\ p_1 c \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_1 E_{10} \\ 0 \\ \beta_1\gamma_1 E_{10} \\ 0 \end{pmatrix}, \quad \boldsymbol{E_2} = \begin{pmatrix} E_2 \\ p_2 c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_2 E_{20} \\ \beta_2\gamma_2 E_{20} \\ 0 \\ 0 \end{pmatrix}$$

The Lorentz transformations that take these to the rest frame of 1,**S′**, and to the rest frame of 2, **S″**, are:

$$\boldsymbol{L_1} = \begin{pmatrix} \gamma_1 & 0 & -\beta_1\gamma_1 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_1\gamma_1 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{L_2} = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 & 0 & 0 \\ -\beta_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

In **S′**, we then have:

$$\boldsymbol{x'_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{x'_2} = \begin{pmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{E'_1} = \begin{pmatrix} E_{10} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{E'_2} = \begin{pmatrix} E'_2 \\ p'_{2x} c \\ p'_{2y} c \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma'_2 E_{20} \\ \beta'_{2x}\gamma'_2 E_{20} \\ \beta'_{2y}\gamma'_2 E_{20} \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_1\gamma_2 E_{20} \\ \beta_2\gamma_2 E_{20} \\ -\beta_1\gamma_1\gamma_2 E_{20} \\ 0 \end{pmatrix}$$

The force on particle 2 is:

$$\boldsymbol{F'_{21}} = -\frac{\tilde{G}E_{10}E'_2}{x'^2_2}\hat{x} = -\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1\gamma_2\hat{x}$$

For the time component of the four-force, we take the dot product of this with $\vec{\beta}'_2$:

$$\boldsymbol{F'_{21}}\cdot\vec{\beta}'_2 = -\frac{\tilde{G}E_{10}E_{20}\gamma_1\gamma_2}{x_2^2}\frac{p'_{2x}c}{E'_2} = -\frac{\tilde{G}E_{10}E_{20}\gamma_1\gamma_2}{x_2^2}\frac{\beta_2}{\gamma_1} = -\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\beta_2\gamma_2$$

Combining these results with $\gamma'_2 = \gamma_1\gamma_2$, we can write the four-force as:

$$\boldsymbol{f}'_{21} = \gamma'_2 \begin{pmatrix} \vec{\beta}'_2 \cdot \boldsymbol{F}'_{21} \\ F'_x \\ F'_y \\ F'_z \end{pmatrix} = -\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1\gamma_2 \begin{pmatrix} \beta_2\gamma_2 \\ \gamma_1\gamma_2 \\ 0 \\ 0 \end{pmatrix}$$

Using the inverse transformation, we boost back to **S**:

$$\boldsymbol{f}_{21} = -\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1\gamma_2 \begin{pmatrix} \gamma_1\beta_2\gamma_2 \\ \gamma_1\gamma_2 \\ \beta_1\gamma_1\beta_2\gamma_2 \\ 0 \end{pmatrix} = -\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1^2\gamma_2^2 \begin{pmatrix} \beta_2 \\ 1 \\ \beta_1\beta_2 \\ 0 \end{pmatrix}$$

Finally, we can deduce the three-force on particle 2 in **S**:

$$\boldsymbol{F}_{21} = -\frac{1}{\gamma_2}\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1^2\gamma_2^2 \begin{pmatrix} 1 \\ \beta_1\beta_2 \\ 0 \end{pmatrix} = -\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1^2\gamma_2 \begin{pmatrix} 1 \\ \beta_1\beta_2 \\ 0 \end{pmatrix}$$

The component of the force in the –*y* direction is the force of gravitational magnetism. Note that it is dependent on the product of the velocities.

Now we want to find the force on particle 1 in the **S** frame for comparison. We proceed in precisely the same fashion above. In **S″**, we have:

$$\boldsymbol{x}''_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}''_2 = \begin{pmatrix} -\beta_2\gamma_2 x_2 \\ \gamma_2 x_2 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{E}''_1 = \begin{pmatrix} E''_1 \\ p''_{1x}c \\ p''_{1y}c \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma''_1 E_{10} \\ \beta''_{1x}\gamma''_1 E_{10} \\ \beta''_{1y}\gamma''_1 E_{10} \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_2\gamma_1 E_{10} \\ -\beta_2\gamma_2\gamma_1 E_{10} \\ \beta_1\gamma_1 E_{10} \\ 0 \end{pmatrix}, \quad \boldsymbol{E}''_2 = \begin{pmatrix} E_{20} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The force on particle 1 is:

$$\boldsymbol{F}'_{12} = +\frac{\tilde{G}E_{20}E''_1}{x''^2_2}\hat{x} = +\frac{\tilde{G}E_{10}E_{20}}{\gamma_2^2 x_2^2}\gamma_1\gamma_2\hat{x} = +\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\frac{\gamma_1}{\gamma_2}\hat{x}$$

For the time component of the four-force, we take the dot product of this with $\vec{\beta}''_1$:

$$\boldsymbol{F}'_{12} \cdot \vec{\beta}'_2 = \frac{\tilde{G}E_{10}E_{20}}{x_2^2}\frac{\gamma_1}{\gamma_2}\frac{p''_{1x}c}{E''_1} = -\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\frac{\gamma_1\beta_2}{\gamma_2}$$

Combining these results with $\gamma''_1 = \gamma_1\gamma_2$, we can write the four-force as:

$$\boldsymbol{f}''_{\mathbf{12}} = \gamma''_1 \begin{pmatrix} \vec{\beta}''_1 \cdot \boldsymbol{F}''_{\mathbf{12}} \\ F''_x \\ F''_y \\ F''_z \end{pmatrix} = \gamma_1\gamma_2 \frac{\tilde{G}E_{10}E_{20}}{x_2^2}\frac{\gamma_1}{\gamma_2}\begin{pmatrix} -\beta_2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1^2 \begin{pmatrix} -\beta_2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We boost back to **S**:

$$\boldsymbol{f}_{\mathbf{12}} = \frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1^2 \begin{pmatrix} -\beta_2\gamma_2 + \beta_2\gamma_2 \\ -\beta_2^2\gamma_2 + \gamma_2 \\ 0 \\ 0 \end{pmatrix} = \frac{\tilde{G}E_{10}E_{20}}{x_2^2}\gamma_1^2 \begin{pmatrix} 0 \\ 1/\gamma_2 \\ 0 \\ 0 \end{pmatrix} = \frac{\tilde{G}E_{10}E_{20}}{x_2^2}\frac{\gamma_1^2}{\gamma_2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

So the three-force on particle 1 in **S**:

$$\boldsymbol{F}_{\mathbf{12}} = \frac{1}{\gamma_1}\frac{\tilde{G}E_{10}E_{20}}{x_2^2}\frac{\gamma_1^2}{\gamma_2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\tilde{G}E_{10}E_{20}}{x_2^2}\frac{\gamma_1}{\gamma_2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Notice that this time there is no force from gravitational magnetism. This is because particle 1 is located along the line of particle 2's motion. If particle 1 were at some other location along the *y* axis, it would experience the effects of gravitational magnetism. Note that the *x* component of the forces are also different because of the different γs that enter each expression. Also, The force on particle 1 from particle 2 is not equal and opposite to the force of particle 2 on particle 1. Newton’s Third Law does not generally hold in relativistic dynamics. Therefore Newton’s Third Law does not hold for magnetic forces, either.

Finally, we note that Newton’s Law of Gravity does not hold, either. In fact, Einstein’s postulate that gravitational mass and inertial mass must be equivalent led to a completely new theory of gravitation, the General Theory of Relativity. General relativity is based on the notion that matter affects the curvature of space and the curvature of space, in turn, affects the motion of matter.

B–4. Some Concluding Thoughts

Many people think of relativity as a set of peculiar rules that govern hypothetical fast-moving objects. But we should recognize relativity as a fundamental theory of space, time, and motion. Its consequences sometimes seem bizarre, because they contradict our everyday experience. But if we do not understand the basics of relativity, we have a flawed picture of natural law, even on an everyday level. Relativity has shown us that mass end energy can be considered to be equivalent in some sense, and allowed us to consider the consequences of transforming mass into energy. Relativity is the cause of everyday magnetic phenomena, and electromagnetic theory can be understood at a fundamental level only through relativistic principles. But perhaps most importantly, special relativity gives us a tantalizing glimpse into the fundamental nature of space and time and how they are intertwined.