Appendix A – Propagating the World Line

We introduced the idea of a propagator as a means of understanding the role of energy in relativistic kinematics. In order to see how the propagator works to generate the world line of an object.

We begin by specifying the initial space-time vector and the initial energy-momentum vector of the object.

$$\mathbf{x}_i = \begin{pmatrix} w_i \\ x_i \\ y_i \\ z_i \end{pmatrix} \qquad \mathbf{E}_i = \begin{pmatrix} E_i \\ p_{xi}c \\ p_{yi}c \\ p_{zi}c \end{pmatrix}$$

In Eq. (2.5), we saw that the change in the space-time vector can be expressed in terms of the propagator as:

$$\begin{pmatrix} \Delta w \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \varepsilon \begin{pmatrix} m\gamma c^2 \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \tag{A–1}$$

ε should be very small, but its exact size is arbitrary. Based on the first component of this vector equation, we can express ε in terms of Δw:

$$\varepsilon = \frac{m\gamma c^2}{\Delta w} = \frac{E}{\Delta w}.$$

The next step is to find how the energy-momentum vector changes during each step. This is given by:

$$\Delta \mathbf{E} = \begin{pmatrix} F_t \\ F_x \\ F_y \\ F_z \end{pmatrix} \Delta w = \begin{pmatrix} \vec{\beta} \cdot \mathbf{F} \\ F_x \\ F_y \\ F_z \end{pmatrix} \varepsilon E \tag{A–2}$$

So then we just step through our algorithm to get the world line of the particle: on the basis of the initial position and energy-momentum vector we step along the energy momentum vector a distance ε to get the next point on the world line, then we use (2–6) to calculate the new energy-momentum vector at the new point, and so forth.

Now let's apply this to an example. An electron with a kinetic energy of 4.00 MeV is traveling in the +*x*-direction. En electric field of 2.00×10^{18} V/m points in the –*y*-direction. Find the trajectory of the electron. The rest energy of an electron is 0.511 MeV and the charge of an electron is -1.602×10^{-19} C.

If you are not familiar with the units, an eV is an electron-volt. This is the kinetic energy gained by an electron (or any particle with a the same charge as an electron) when it is accelerated through a potential of one volt. The conversion factor to SI units is 1 eV = 1.602 × 10^{-19} J.

The force on a charge in an electric field is just $F = qE_{el}$ where E_{el} is the electric field strength. The direction of the force is in the direction of the field for positively charged particles and opposite the field for negatively charged particles.

Let's first find a few important quantities:

Total energy: $E = K + E_0 = 4.51\ \text{MeV}$

γ: $$\gamma = \frac{E}{E_o} = \frac{4.51}{0.511} = 8.83$$

β: $$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.994$$

The initial **E**: $$\mathbf{E} = \begin{pmatrix} E \\ \beta E \\ 0 \\ 0 \end{pmatrix}$$

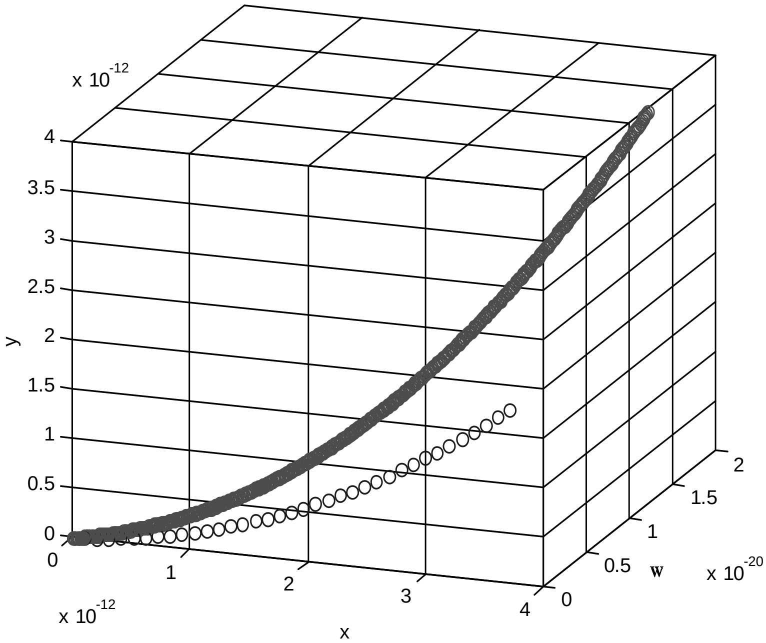

Figure A–1 The world line of a highly relativistic electron in a constant electric field. The corresponding non-relativistic result is shown by open circles.

The change in **E**: $$\Delta\mathbf{E} = \begin{pmatrix} \beta\cdot\mathbf{F} \\ F_x \\ F_y \\ F_z \end{pmatrix} \varepsilon E = \begin{pmatrix} \beta_y e E_{el} E \\ 0 \\ e E_{el} E \\ 0 \end{pmatrix} \varepsilon = \begin{pmatrix} p_y c \\ 0 \\ E \\ 0 \end{pmatrix} \varepsilon e E_{el}$$

Now we just continue the process step by step with the change in **x** being ε **E** . This gives us the results shown in Figure A–1. In the relativistic case, the mass is larger; however, in the non-relativistic case, the speed of the electrons is considerably larger than the speed of light, causing the force to have a smaller effect on the trajectory of the particle.