

Chapter 4 – Consequences of the Lorentz Transformation in Space and Time

4-1. The Speed of Light

Now that we have the Lorentz Transformation, we have the basic tool that is necessary to understand physical law at a more fundamental level than Newtonian mechanics permits. We will now apply the Lorentz Transformation to a variety of specific examples. In each of these examples, we will have two reference frames, \mathbf{S} and \mathbf{S}' . Typically \mathbf{S}' moves at a velocity v in the $+z$ direction with respect to \mathbf{S} . Quantities measured in the \mathbf{S}' frame will be denoted with primes. These frames are synchronized in the normal fashion; that is, the origins of \mathbf{S} and \mathbf{S}' are in the same location in space at time $t=0$. We will be careful to define events so that we can unambiguously determine space-time coordinates in each inertial frame.

Example 1: In \mathbf{S} , a pulse of light is emitted from the origin at $t=0$. The pulse is detected at coordinates x, y, z, w a short time later.

	Event 1	Event 2
In \mathbf{S}	$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\mathbf{r}_2 = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$
In \mathbf{S}'	$\mathbf{x}'_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\mathbf{x}'_2 = \begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix}$

Note that the velocity of the pulse in \mathbf{S} is:

$$c = \frac{\Delta r}{\Delta t} = \frac{\sqrt{x^2 + y^2 + z^2}}{t}$$

$$w^2 - x^2 - y^2 - z^2 = 0$$

In \mathbf{S}' the pulse is also emitted at the origin at $t'=0$, since that is the way the two inertial frames are synchronized. Now, let us find the coordinates of the point where light is detected in \mathbf{S}' .

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma w - \beta\gamma z \\ x \\ y \\ \gamma z - \beta\gamma w \end{pmatrix}$$

Note that the time at which the pulse is detected in \mathbf{S}' is not the same as in \mathbf{S} . Now, in order to determine the speed of light in \mathbf{S}' , let us do a little algebra:

$$\begin{aligned}
& w'^2 - x'^2 - y'^2 - z'^2 \\
&= (\gamma w - \beta \gamma z)^2 - x'^2 - y'^2 - (\gamma z - \beta \gamma w)^2 \\
&= \gamma^2 w^2 - 2\beta \gamma^2 w z + \beta^2 \gamma^2 z^2 - x'^2 - y'^2 - \gamma^2 z^2 + 2\beta \gamma^2 w z - \beta^2 \gamma^2 w^2 \\
&= \gamma^2 w^2 + \beta^2 \gamma^2 z^2 - x'^2 - y'^2 - \gamma^2 z^2 - \beta^2 \gamma^2 w^2 \\
&= w^2(\gamma^2 - \beta^2 \gamma^2) - x'^2 - y'^2 - z^2(\gamma^2 - \beta^2 \gamma^2) \\
&= w^2 - x'^2 - y'^2 - z^2 \\
&= 0
\end{aligned}$$

From this it is clear that the speed of the light pulse in \mathbf{S}' is also c . Since this derivation is perfectly general (that is, we can call the direction of any boost the $+z$ direction, and we let the pulse of light travel in a completely arbitrary direction), we may conclude the following important result:

The speed of light in vacuum measured with respect to any observer is always c , irrespective of the motion of the source or the observer.

This fact is called the “Second Postulate of Relativity.” Along with Galilean Relativity, this formed the foundation of Einstein’s derivation of the Lorentz Transformations. Since we have begun with kinematic considerations, however, it is a necessary consequence of our postulate, $\Delta E = \Delta K$.

Note also that, since the speed of light is a constant in all directions, the wavefront of the light is a sphere in any inertial reference frame.

4-2. Relativistic Invariants

In Newtonian mechanics, the speed of light is not a constant in all reference frames; however, some quantities are the same in all inertial frames. These quantities include the time between two events and the distance between two events. Such quantities are called Newtonian (or Galilean) “invariants.” It is useful to ask what quantities are relativistic (or Lorentz) invariants.

Before we answer that question, however, we need to introduce a new matrix called the “metric tensor.” In special relativity, the metric tensor is a simple 4×4 matrix:

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that the metric tensor looks a lot like the identity matrix, except that three of the diagonal elements are -1 . Now we define the “inner product” or “dot product” in space-time as

$$\mathbf{a}^4 \cdot \mathbf{b}^4 = \mathbf{a}^{4T} \mathbf{g} \mathbf{b}^4 = \begin{pmatrix} a_t & a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} b_t \\ b_x \\ b_y \\ b_z \end{pmatrix} = a_t b_t - a_x b_x - a_y b_y - a_z b_z = a_t b_t - \mathbf{a}^3 \cdot \mathbf{b}^3$$

Note that this is the product of the two time components of the four-vectors minus the usual dot product in three

dimensions. Let us now see how this transforms between reference frames. We will need to make use of the matrix identity $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$.

$$\mathbf{a}' \cdot \mathbf{b}' = \mathbf{a}'^T \mathbf{g} \mathbf{b}' = (\mathbf{L}\mathbf{a})^T \mathbf{g} \mathbf{L}\mathbf{b} = \mathbf{a}^T \mathbf{L}^T \mathbf{g} \mathbf{L}\mathbf{b} = \mathbf{a}^T \mathbf{L} \mathbf{g} \mathbf{L}\mathbf{b} = \mathbf{a}^T \mathbf{g} \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$$

We have made use of the identity $\mathbf{L} \mathbf{g} \mathbf{L} = \mathbf{g}$. We now prove it for a boost in the +z direction; however, the results can be generalized to any boost.

$$\begin{aligned} \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} &= \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \beta\gamma & 0 & 0 & -\gamma \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\gamma^2(1-\beta^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

The conclusion is a very useful result:

Any inner product is a relativistic invariant. That is, it has the same value in any inertial reference frame.

To see how these invariants work, let's do two examples. First, we look at the four-momentum.

$$\mathbf{E} \cdot \mathbf{E} = E^2 - p_x^2 c^2 - p_y^2 c^2 - p_z^2 c^2 = E^2 - p^2 c^2 = E_0^2$$

The invariant associated with this vector is just the square of the rest energy. This clearly has the same value in all reference frames. Second, we take the inner product of the four-momentum with the four-position of a particle moving in the x direction.

$$\mathbf{E} \cdot \mathbf{x} = Ew - pcx.$$

This quantity is not obviously invariant, although mathematically we know that it must be. The meaning of this quantity becomes more clear if we use the quantum mechanical relationships that relate wave and particle aspects of a particle: Einstein's relationship, $E = \hbar\omega$, and de Broglie's relationship, $p = \hbar k$, where \hbar is Planck's constant divided by 2π . The expression then becomes:

$$\mathbf{E} \cdot \mathbf{x} = \hbar\omega ct - \hbar kcx = -\hbar c(kx - \omega t).$$

This, then, is a constant multiplied by the phase of a wave. Thus, if one observer sees the crest of a wave at a certain point in space and time, all other observers in different inertial frames would also recognize it to be crest.

4-3. Space and Time

We have already seen that time intervals are not measured to be the same by two different observers, but that the speed of light is a constant. This implies that different observers also measure distances between two points differently. To see how space and time behave with Lorentz transformation, we now proceed to do three specific examples.

Example 2. A rod is at rest in S' . The rod lies along the z' axis with one end at the origin. What is the length of the rod according to an observer in S ?

We may think of the measurement of each end of the rod as one event – as we make the measurement we set off a little explosion. For convenience, we can measure the end of the rod at the origin at $t' = 0$. In S' we may measure the other end of the rod at any time we choose. As we consider the measurement in S , however, we need to measure both ends of the rod at the same instant because the rod is in motion. Since we measure the end at the origin at $t=0$, we must measure the opposite end at $t=0$ as well.

	Event 1	Event 2
In S	$x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$r_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ z \end{pmatrix}$
In S'	$x'_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$x'_2 = \begin{pmatrix} w' \\ 0 \\ 0 \\ z' \end{pmatrix}$

$$x'_2 = L x_2, \quad \begin{pmatrix} w' \\ 0 \\ 0 \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} -\beta\gamma z \\ 0 \\ 0 \\ \gamma z \end{pmatrix}$$

So, if a rod is moving at a velocity v along the direction of its length, its length (z) is its rest length (z') divided by γ . That is:

Moving rods are contracted by a factor of γ along their direction of motion.

It should also be noted that event 1 is simultaneous with event 2 in S , but not in S' .

Simultaneity is relative.

Example 3. A rod is at rest in S' . The rod lies along the x' axis with one end at the origin. What is the length of the rod according to an observer in S ?

We proceed as in the previous example.

	Event 1	Event 2
In \mathbf{S}	$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\mathbf{x}_2 = \begin{pmatrix} 0 \\ x \\ 0 \\ 0 \end{pmatrix}$
In \mathbf{S}'	$\mathbf{x}'_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\mathbf{x}'_2 = \begin{pmatrix} w' \\ x' \\ 0 \\ 0 \end{pmatrix}$

$$\mathbf{x}'_2 = L \mathbf{x}_2, \quad \begin{pmatrix} ct' \\ x' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x \\ 0 \\ 0 \end{pmatrix}$$

If a rod is moving at a velocity \mathbf{v} perpendicular to the direction of its length, its length (x) is its rest length (x'). That is:

The length of moving rods remains unchanged perpendicular to their direction of motion.

Example 4. A clock is at rest in \mathbf{S}' . The clock is placed at the origin. When the clock reads t' in \mathbf{S}' , what does it read in \mathbf{S} ?

We can think of two ticks of the clock as the events of interest. For simplicity, we let the time of the first tick be $t' = 0$. We then have:

	Event 1	Event 2
In \mathbf{S}	$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\mathbf{x}_2 = \begin{pmatrix} w \\ 0 \\ 0 \\ z \end{pmatrix}$

In S'	$x'_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$x'_2 = \begin{pmatrix} w' \\ 0 \\ 0 \\ 0 \end{pmatrix}$
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$$x_2 = L^{-1} x'_2, \quad \begin{pmatrix} ct \\ 0' \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} w' \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma w' \\ 0 \\ 0 \\ \beta\gamma w' \end{pmatrix}$$

If a clock is moving at a velocity v , its time (t) is its rest time (t') times γ . That is:

Moving clocks run slowly by a factor of γ .