

Chapter 3 – The Lorentz Transformation

3–1. Galilean Relativity and Reference Frames

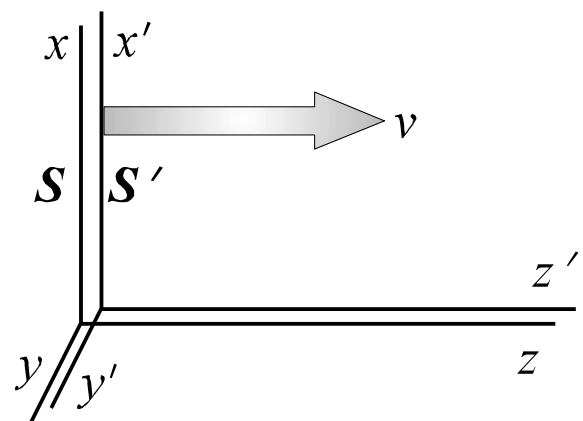
When we formulate the laws of physics, we either implicitly or explicitly define an observer who makes the measurement. Often times, we would like to know what a different observer measures. For example, when we do collision problems, we often ask how an observer in the “center of mass” or more precisely “zero momentum” system views the collision, because the mathematics is much easier in this system. Since the time of Newton, physicists have accepted the assertion that the laws of physics should have the same mathematical form to any observer who is not experiencing an acceleration. This concept of “Galilean Relativity” was accepted by Albert Einstein as his “First Postulate of Relativity.” In classical physics, the positions and times of an event as measured by two observers are related by “non-relativistic boosts” as described in Section 1.3.C. When we apply a non-relativistic boost to transform coordinates from one observer to another, we say we do a “Galilean Transformation” of the coordinates. A similar relativistic transformation is called a “Lorentz Transformation,” named after the great Dutch physicist, H. A. Lorentz. Lorentz proposed these transformations, which are central to relativistic theory, a year before Einstein published his Special Theory of Relativity.

Before we find the Lorentz boost matrix, however, we should be careful to define a few very useful terms.

- *event*: an occurrence which has definite coordinates in both space and time. It is often useful to think of an event as a little explosion.
- *reference frame*: a set of measuring rods and clocks. The rods are arranged along Cartesian axes from an origin. Clocks are located at every point in space. The clocks are synchronized so that they all read the same time. This means that any two observers equidistant from an event say that light from the event arrives at their clocks at the same time.
- *inertial reference frame*: a reference frame that is not experiencing acceleration. If a reference frame is accelerating, then the laws of physics must take that into consideration. For example, a ball thrown straight up might appear to veer to one side.
- *synchronizing two reference frames*: to easily relate coordinates in one reference frame with another moving with respect to it, at time zero we put the origins of both frames at the same location with the x , y , and z axes pointing in the same directions. That is, we let some event define $(x, y, z, ct) = (x', y', z', ct') = (0, 0, 0, 0)$.

3–2. Deriving the Lorentz Transformation for a Boost in the z Direction.

Based on the motion of a particle as viewed in two different inertial frames, we can deduce what the properties of the Lorentz transformation must be. Let us take an inertial reference frame, S , and a second inertial frame, S' , moving in the $+z$ direction with respect to S at a velocity $v = \beta c$. In order for Galilean Relativity to be satisfied, we need to assume a linear relationship between coordinates in these frames. Therefore we can write the transformation as a matrix operator that takes an energy-momentum four-vector from S into S' . We write this matrix with components L_{ij} . We will fill in elements of the transformation matrix one step at a time.



1. A particle at rest in \mathbf{S} will appear to move in the $-z$ direction at a velocity v in \mathbf{S}' .

$$\mathbf{E}' = \begin{pmatrix} L_{tt} & L_{tx} & L_{ty} & L_{tz} \\ L_{xt} & L_{xx} & L_{xy} & L_{xz} \\ L_{yt} & L_{yx} & L_{yy} & L_{yz} \\ L_{zt} & L_{zx} & L_{zy} & L_{zz} \end{pmatrix} \begin{pmatrix} E_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_{tt}E_0 \\ L_{xt}E_0 \\ L_{yt}E_0 \\ L_{zt}E_0 \end{pmatrix} = \begin{pmatrix} \gamma E_0 \\ 0 \\ 0 \\ -\beta \gamma E_0 \end{pmatrix}$$

$$\mathbf{L}(\beta \hat{z}) = \begin{pmatrix} \gamma & L_{tx} & L_{ty} & L_{tz} \\ 0 & L_{xx} & L_{xy} & L_{xz} \\ 0 & L_{yx} & L_{yy} & L_{yz} \\ -\beta \gamma & L_{zx} & L_{zy} & L_{zz} \end{pmatrix}$$

2. A particle moving in \mathbf{S} with velocity v in the $+z$ direction will appear to be at rest in \mathbf{S}' .

$$\begin{pmatrix} \gamma & L_{tx} & L_{ty} & L_{tz} \\ 0 & L_{xx} & L_{xy} & L_{xz} \\ 0 & L_{yx} & L_{yy} & L_{yz} \\ -\beta \gamma & L_{zx} & L_{zy} & L_{zz} \end{pmatrix} \begin{pmatrix} \gamma E_0 \\ 0 \\ 0 \\ \beta \gamma E_0 \end{pmatrix} = \begin{pmatrix} \gamma^2 E_0 + L_{tz} \beta \gamma E_0 \\ L_{xz} \beta \gamma E_0 \\ L_{yz} \beta \gamma E_0 \\ -\beta \gamma^2 E_0 + L_{zz} \beta \gamma E_0 \end{pmatrix} = \begin{pmatrix} E_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The first equation requires a little algebra. First, Let us derive a very useful identity:

$$\begin{aligned} \gamma^2 &= \frac{1}{1 - \beta^2} \\ \gamma^2 - \beta^2 \gamma^2 &= 1 \\ \beta^2 \gamma^2 &= \gamma^2 - 1 \end{aligned}$$

Now we return to the first equation:

$$\begin{aligned} \gamma^2 E_0 + L_{tz} \beta \gamma E_0 &= E_0 \\ \gamma^2 + L_{tz} \beta \gamma &= 1 \\ -L_{tz} \beta \gamma &= \gamma^2 - 1 = \beta^2 \gamma^2 \\ L_{tz} &= -\beta \gamma \end{aligned}$$

Thus, the transformation equation now becomes:

$$\mathbf{L}(\beta \hat{z}) = \begin{pmatrix} \gamma & L_{tx} & L_{ty} & -\beta \gamma \\ 0 & L_{xx} & L_{xy} & 0 \\ 0 & L_{yx} & L_{yy} & 0 \\ -\beta \gamma & L_{zx} & L_{zy} & \gamma \end{pmatrix}$$

3. Now we need to add a component in the x or y direction to get some information on the remaining components of \mathbf{L} . Intuitively, we might suggest that observers in \mathbf{S} and \mathbf{S}' would measure the same thing for the x component of velocity and momentum. Unfortunately, we have a paradox here. If both observers measure the same x component of velocity, they still measure different total velocities, and because they measure different total velocities, they measure different x components of momentum. Conversely, if they measure the same component of momentum, they must measure different components of velocity. It is clear we must proceed carefully or we might be deceived by our intuition. We do know, though, that if a particle has a velocity $\mathbf{v} = v_x \hat{x} + v_z \hat{z}$ in \mathbf{S} , the velocity must only have an x component in \mathbf{S}' .

$$\begin{pmatrix} \gamma & L_{tx} & L_{ty} & -\beta\gamma \\ 0 & L_{xx} & L_{xy} & 0 \\ 0 & L_{yx} & L_{yy} & 0 \\ -\beta\gamma & L_{zx} & L_{zy} & \gamma \end{pmatrix} \begin{pmatrix} \gamma_1 E_0 \\ \beta_x \gamma_1 E_0 \\ 0 \\ \beta \gamma_1 E_0 \end{pmatrix} = \begin{pmatrix} \gamma \gamma_1 E_0 + L_{tx} \beta_x \gamma_1 E_0 - \beta^2 \gamma \gamma_1 E_0 \\ L_{xx} \beta_x \gamma_1 E_0 \\ L_{yx} \beta_x \gamma_1 E_0 \\ -\beta \gamma \gamma_1 E_0 + L_{zx} \beta_x \gamma_1 E_0 + \beta \gamma \gamma_1 E_0 \end{pmatrix} = \begin{pmatrix} \gamma'_1 E_0 \\ \beta'_x \gamma'_1 E_0 \\ 0 \\ 0 \end{pmatrix}$$

Note that $L_{yx} = 0$ and that, since the first and third terms in the last line cancel, $L_{zx} = 0$. We will ignore the top two equations for now. However, by a similar argument with a y component in the four-vectors, we can deduce that $L_{xy} = L_{zy} = 0$. This leaves us with:

$$\mathbf{L}(\beta \hat{z}) = \begin{pmatrix} \gamma & L_{tx} & L_{ty} & -\beta\gamma \\ 0 & L_{xx} & 0 & 0 \\ 0 & 0 & L_{yy} & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

4. At this point, we note that $\mathbf{L}(-\beta \hat{z}) = \mathbf{L}^{-1}(+\beta \hat{z})$. This then implies:

$$\mathbf{L}(+\beta \hat{z}) \mathbf{L}(-\beta \hat{z}) = \begin{pmatrix} \gamma & L_{tx} & L_{ty} & -\beta\gamma \\ 0 & L_{xx} & 0 & 0 \\ 0 & 0 & L_{yy} & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & \ell_{tx} & \ell_{ty} & +\beta\gamma \\ 0 & \ell_{xx} & 0 & 0 \\ 0 & 0 & \ell_{yy} & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix} = \mathbf{I}$$

where the ℓ s are similar to the L s but with the signs of β reversed. Multiplying this out, we have:

$$\begin{pmatrix} \gamma & L_{tx} & L_{ty} & -\beta\gamma \\ 0 & L_{xx} & 0 & 0 \\ 0 & 0 & L_{yy} & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & \ell_{tx} & \ell_{ty} & +\beta\gamma \\ 0 & \ell_{xx} & 0 & 0 \\ 0 & 0 & \ell_{yy} & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \gamma^2(1-\beta^2) & \gamma \ell_{tx} + L_{tx} \ell_{xx} & \gamma \ell_{ty} + L_{ty} \ell_{yy} & \beta \gamma^2 - \beta \gamma^2 \\ 0 & L_{xx} \ell_{xx} & 0 & 0 \\ 0 & 0 & L_{yy} \ell_{yy} & 0 \\ -\beta \gamma^2 + \beta \gamma^2 & -\beta \gamma \ell_{tx} & -\beta \gamma \ell_{ty} & \gamma^2(1-\beta^2) \end{pmatrix} \\ = \begin{pmatrix} 1 & \gamma \ell_{tx} + L_{tx} \ell_{xx} & \gamma \ell_{ty} + L_{ty} \ell_{yy} & 0 \\ 0 & L_{xx} \ell_{xx} & 0 & 0 \\ 0 & 0 & L_{yy} \ell_{yy} & 0 \\ 0 & -\beta \gamma \ell_{tx} & -\beta \gamma \ell_{ty} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The bottom lines require ℓ_{tx} and ℓ_{ty} to be zero. The middle two lines, however, indicate that L_{xx} , L_{yy} , ℓ_{xx} , and ℓ_{yy} can *not* be zero. With these results, we can conclude from the first line that L_{tx} and L_{ty} must also be zero. Finally, we know that L_{xx} and ℓ_{xx} must be identical except for the sign of β . Since the product $L_{xx} \ell_{xx} = 1$, We may conclude $L_{xx} = \ell_{xx} = 1$.

We therefore have the mathematical form of a Lorentz boost in the $+z$ direction:

$$\mathbf{L}(\beta\hat{z}) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \quad (3-1)$$

Similar arguments lead to boosts along the other axes:

$$\mathbf{L}(\beta\hat{x}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{L}(\beta\hat{y}) = \begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3-2)$$

Finally, we can take the boost along the z axis and rotate the transformation matrix as described in Chapter 1 to obtain a boost in a general direction. The result is:

$$\mathbf{L}(\boldsymbol{\beta}) = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + \frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1 + \frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1 + \frac{(\gamma-1)\beta_z^2}{\beta^2} \end{pmatrix} \quad (3-3)$$

3-3 Lorentz Transformations for Space-Time Vectors

So far we have established transformation matrices for energy-momentum vectors. The next thing we'd like to know is how space-time vectors (four-positions) transform. The answer to this is easy because we know that the four-momentum is the propagator for a particle's world line. Let's make the argument a bit more concrete. Assume that \mathbf{S} and \mathbf{S}' are synchronized according to the description on page 3-1. Let event 1 be the collision of a particle with a second particle. We let this collision take place at the origin of \mathbf{S} at time $t = 0$. Let event 2 be the collision of the same particle with a third particle. This takes place at some four-position \mathbf{x} . Since the momentum of the particle remains constant between the collisions, we know that \mathbf{x} is just some constant times the four-momentum. That is:

$$\text{In } \mathbf{S}: \quad \mathbf{x} = \alpha \mathbf{E}$$

$$\text{In } \mathbf{S}': \quad \mathbf{x}' = \alpha \mathbf{E}' = \alpha \mathbf{L} \mathbf{E} = \mathbf{L} \alpha \mathbf{E} = \mathbf{L} \mathbf{x}$$

Thus, the same transformation matrix works for four-positions and four-momenta. Any quantity that transforms using the standard Lorentz transformation matrices is called a "four-vector."

3-4 The Relativistic Doppler Shift

As an example of applying the Lorentz transformations, we will transform the four-momenta of photons. To do that, however, we need one more concept. Since a photon is massless and can not ever be at rest, it doesn't make sense to speak of its rest energy. On the other hand, Einstein suggested, also in 1905, that light comes in "packets" called photons having an energy equal to Planck's constant times the frequency of the light. That is:

$$E = hf \quad (3-4)$$

where $h = 6.626 \times 10^{-34}$ Js. Then, by the relativistic energy-momentum relationship, $pc = E$. From this we can construct a four-vector and boost the four-vector to an arbitrary frame.

Let us begin by taking a photon moving in the $-z$ direction. If it is seen with a frequency f in frame \mathbf{S} , then what frequency is measured in a frame \mathbf{S}' moving in the $+z$ direction with a velocity v ?

$$\begin{pmatrix} hf' \\ 0 \\ 0 \\ -hf' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} hf \\ 0 \\ 0 \\ -hf \end{pmatrix} = \begin{pmatrix} \gamma hf + \beta\gamma hf \\ 0 \\ 0 \\ -\gamma hf - \beta\gamma hf \end{pmatrix}$$

This gives $f' = \gamma(1+\beta)f$. Note that if β is positive, the frequency increases and if β is large and negative, the frequency decreases, as expected. But if $\gamma = 1/(1-\beta)$, which is possible if β is negative, there is no Doppler shift at all.

Classically, if an observer is moving transverse to a source, so the source and observer are neither moving toward nor away from each other, there is no Doppler shift observed. The time it takes for one wave to pass two such observers is the same. Now, let's see what happens relativistically.

$$\begin{pmatrix} hf' \\ hf' \sin\theta \\ 0 \\ hf' \cos\theta \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} hf \\ hf \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma hf \\ hf \\ 0 \\ -\beta\gamma hf \end{pmatrix}$$

The transverse relativistic Doppler shift is then $f' = \gamma f$. Relativistically, it takes different times for the wave to pass two observers moving at different speeds transverse to the wave. The reason for this will become clear in the next chapter.

3-5 Relativistic Velocity Transformation

We noted above that momentum and velocity must transform according to two different sets of rules. As we can see from the Lorentz transformation Eq. (3-1), the momentum transforms quite simply. From this, we should be able to derive rules for velocity transformation as well.

Let us first take a particle moving in an arbitrary direction and then take a boost in the z direction:

$$\begin{pmatrix} E' \\ 0 \\ 0 \\ p'_z/c \end{pmatrix} = \begin{pmatrix} \gamma_b & 0 & 0 & -\beta_b \gamma_b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_b \gamma_b & 0 & 0 & \gamma_b \end{pmatrix} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_x c \end{pmatrix} = \begin{pmatrix} \gamma_b E - \beta_b \gamma_b p_z c \\ p_x c \\ p_y c \\ \gamma_b p_z c - \beta_b \gamma_b E \end{pmatrix}$$

We have to be careful to remember that the β s and γ s embedded in E and E' are not the same as the β_b and γ_b that appear in the boost matrix. To see how velocity transforms, we need to do a bit of algebra. First we note that:

$$pc = E_0 \boldsymbol{\beta} \gamma, \text{ and } E = E_0 \gamma.$$

$$\text{Therefore } \boldsymbol{\beta} = \frac{pc}{E}$$

In the case above, this becomes:

$$\beta'_z = \frac{\gamma_b p_z c - \beta_b \gamma_b E}{\gamma_b E - \beta_b \gamma_b p_z c} = \frac{p_z c - \beta_b E}{E - \beta_b p_z c} = \frac{\frac{p_z c}{E} - \beta_b}{1 - \beta_b \frac{p_z c}{E}} = \frac{\beta_z - \beta_b}{1 - \beta_b \beta_z}$$

$$\beta'_x = \frac{p_x c}{\gamma_b E - \beta_b \gamma_b p_z c} = \frac{\frac{p_x c}{E}}{\gamma_b - \beta_b \gamma_b \frac{p_z c}{E}} = \frac{\beta_x}{\gamma_b (1 - \beta_b \beta_z)}$$

A similar equation holds for the y direction.

Although relationships between velocities are easily and generally handled in this fashion, many texts prefer to summarize these results in terms of “velocity addition” formulas. These are obtained by rewriting the above relationships for the case that the boost is in the $-\hat{z}$ direction so that the velocity of the object increases by β_b in the $+\hat{z}$ direction. These equations are:

$$\begin{aligned} \beta'_\parallel &= \frac{\beta_\parallel + \beta_b}{1 + \beta_\parallel \beta_b} & \beta_\parallel \text{ is the component of } \beta \text{ parallel to } \beta_b \\ \beta'_\perp &= \frac{\beta_\perp}{\gamma_b (1 + \beta_\parallel \beta_b)} & \beta_\perp \text{ is any component of } \beta \text{ perpendicular to } \beta_b \end{aligned} \tag{3-5}$$