

Review Topics

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R.1 Vectors

I assume that you have already studied vectors in previous physics courses. If you have not been introduced to vectors yet, you should look over the section on vectors in any standard calculus-based physics text.

A. Geometric Model of Vectors

In your previous physics courses, you have had to deal with a number of vector quantities including displacement, velocity, acceleration, and force.¹ We usually define a vector as a quantity that has both magnitude and direction. This is in contrast with a scalar which has magnitude but no direction. When you think of a vector, you probably think of an arrow. An arrow is a good geometrical way to model a vector as the magnitude can correspond to the length of an arrow and the direction to the direction the arrow points. Of course, a vector is not really an arrow; but an arrow is a good way of visualizing a vector.

Let's consider how vectors and vector operations are defined when we think of vectors as arrows.

Magnitude and direction. As an arrow in space, a vector can exist without defining any coordinate system. We merely have to somehow specify the length and direction of the arrow. For example, we could say that a vector is 2.54 cm long and points in the direction of Polaris. As long as we ignore the fact that the direction of Polaris differs slightly from location to location on the earth, this description characterizes the vector. However, we could have one vector on one side of your room of length 2.54 cm pointing toward Polaris and another, different vector, on the other side of the room also 2.54 cm long and pointing toward Polaris. The two vectors are not really the same vector; however, we define the vectors to be equal as long as their magnitudes and directions are equal.

If we wish to introduce a coordinate system, we can also define the direction of the vector with respect to that coordinate system. In two dimensions we usually set up an x axis and a y axis perpendicular to the x axis. We specify the vector by two independent numbers: the vector's length and its angle. We define the angle θ to be the direction of the vector measured with respect to the x direction. Note that the vector need not be located at the origin of the coordinate system for θ to be defined.

In three dimensions, we need three independent numbers to specify the direction of a vector. These numbers are the length of the vector and two angles. In spherical coordinates, we usually label these angles as θ and ϕ . In this book, as in most physics texts, the angle θ is defined as the angle of the vector with respect to the z axis. We really won't worry about ϕ . (Note that math texts usually define θ and ϕ oppositely to physics texts.)

Notation. In this book, we denote vectors with an arrow over the top (e.g., \vec{A}) and the magnitude of vectors in italic type (e.g., A).

Addition. To geometrically add two arrows, we place them head to tail and draw the sum as the arrow which extends from the head of the first vector to the tail of the second. Note that it doesn't matter if we move a vector around to a new, convenient location as long as we don't rotate it in the process.

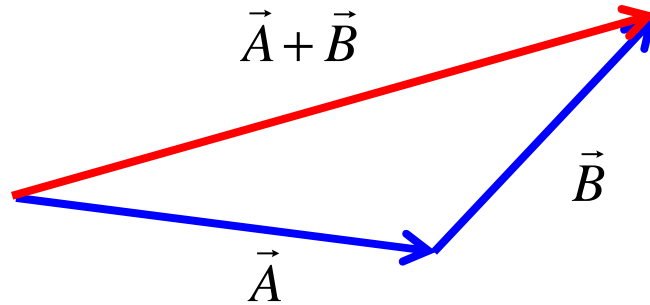


Figure R.1. Adding two vectors geometrically.

Dot Product. There are two methods of multiplying vectors. The first is called the "dot product" because it is represented by the \cdot symbol. It is also called the scalar product because the result is a scalar rather than another vector. The dot product of two vectors \vec{A} and \vec{B} is the product of A (the magnitude) and the component of \vec{B} which is parallel to \vec{A} . That is,

(R.1 Dot product)
$$\vec{A} \cdot \vec{B} = AB \cos \theta$$

where θ is the angle between the vectors when placed tail to tail.

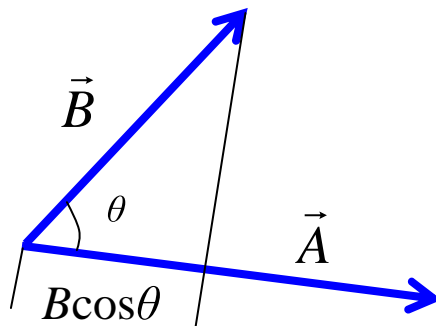


Figure R.2. The dot product of two vectors.

You can convince yourself that the order in which two vectors appear in a dot product is unimportant. That is:

(R.2 Dot product commutes)

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

When the order of a product does not affect the outcome, we say the product commutes.

Cross Product. The second method of multiplying two vectors is called the cross product, represented by \times . It is also called the vector product, as the result is a vector. The magnitude of the cross product of two vectors \vec{A} and \vec{B} is the product of A and the component of \vec{B} that is perpendicular to \vec{A} , that is

(R.3 Cross product)

$$|\vec{A} \times \vec{B}| = AB \sin \theta.$$

Note that this is just the formula for the area of the parallelogram formed by two vectors \vec{A} and \vec{B} , as you can see from Fig. R.3.

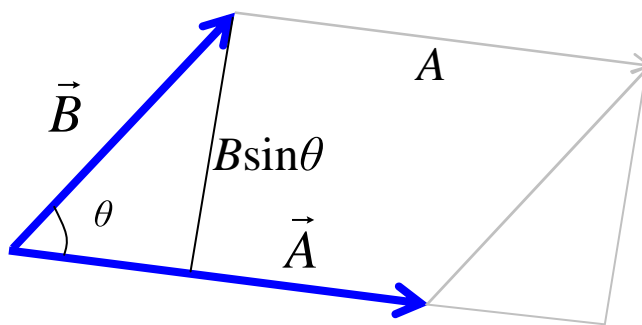


Figure R.3. The cross product of two vectors.

What we want to do now is define the direction of the cross product. If you think about it, it's true that when we place any two vectors tail to tail, they must lie in the same plane. Let's take the screen (or page) as the plane in which \vec{A} and \vec{B} lie. Now let's give you an important rule.

First Rule of Cross Product Direction

The cross product of \vec{A} and \vec{B} is perpendicular to the plane formed when we put the tails of \vec{A} and \vec{B} together. This means that the cross product is perpendicular to both \vec{A} and \vec{B} .

Therefore, in Fig. R.3 the cross product points either into the screen or out of the screen, as these are the only directions perpendicular to the plane of the screen. What remains is for us to determine in which of these two directions the cross product points. In order to remove this ambiguity, a right-hand rule is used. Hold your hand out flat with your fingers pointing in the direction of \vec{A} . Turn your hand so that your fingers rotate toward \vec{B} when you close your hand.

The cross product is in the direction of your thumb. Remember that the right-hand rule will only tell you which of the two possible directions the cross product is in.

By using the right hand rule, you can easily see that if $\vec{A} \times \vec{B}$ is out of the screen as in Fig. R.3, then $\vec{B} \times \vec{A}$ is into the screen. Hence:

(R.4 Cross product anticommutes)
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

When reversing the order of a product only changes the sign of the product, we say the product anticommutes.

B. Algebraic or Component Model of Vectors

Unit vectors. A unit vector is a vector that has a magnitude of one unit. In Cartesian coordinates, we will denote the unit vectors along the x , y , and z axes as \hat{x} , \hat{y} , and \hat{z} respectively. In polar coordinates, we write a unit vector directed outward from the origin (note that the angle is not specified, it can be in any direction directly outward from the origin) as \hat{r} . The unit vector perpendicular to \hat{r} in a counterclockwise sense is $\hat{\theta}$. Similarly, in spherical coordinates the unit vector directed away from the origin is also called \hat{r} . It is important to remember that the magnitude of a unit vector is always one.

Components. In Cartesian coordinates, any vector can be expressed as the sum of vectors in the \hat{x} , \hat{y} , and \hat{z} directions.

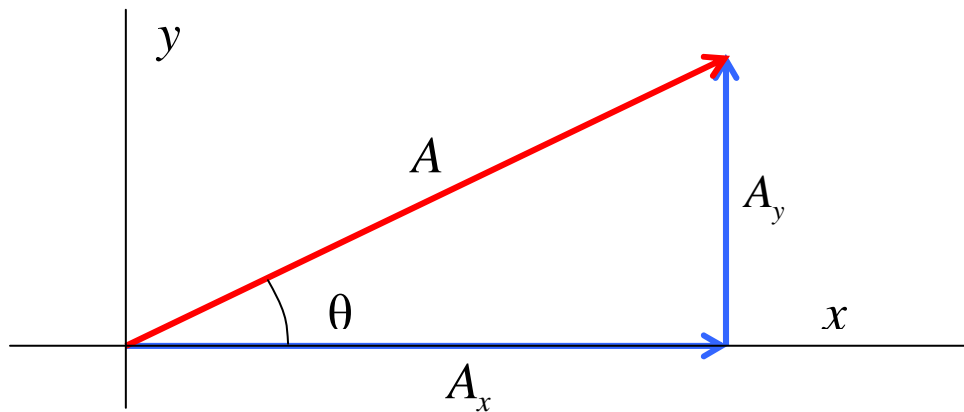


Fig. R.4. The components of a vector.

The vectors that lie in the x and y directions are the components of the vector. We could write $\vec{A} = \vec{A}_x + \vec{A}_y$, but the usual notation uses the x and y unit vectors explicitly.

(R.5 Components of a vector)
$$\vec{A} = A_x \hat{x} + A_y \hat{y}$$

In two dimensions we have some simple relations between the components of a vector and its magnitude and direction:

$$\begin{aligned}
 A_x &= A \cos \theta \\
 A_y &= A \sin \theta \\
 A &= \sqrt{A_x^2 + A_y^2} \\
 \tan \theta &= \frac{A_y}{A_x}
 \end{aligned}$$

(R.6 Component relationships)

Addition. To add vectors, we just add components:

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

(R.7 Vector addition)

Dot product.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

(R.8 Dot product)

Cross product.

$$\begin{aligned}
 \vec{A} \times \vec{B} &= (A_y B_z - A_z B_y)\hat{x} + \\
 &(A_z B_x - A_x B_z)\hat{y} + \\
 &(A_x B_y - A_y B_x)\hat{z}
 \end{aligned}$$

(R.9 Cross product)

Note that if we replace x with y , y with z , and z with x in the first term, then we get the second term. Doing this to the second term gives us the third term. If you are acquainted with determinants, an easy way to remember this form is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(R.10 Cross product)

1 – You also were introduced to a few “axial vectors” or “pseudovectors” such as torque and angular momentum. These are quantities that are similar to vectors in the way they transform under rotations, but differ in the way they transform under reflections. In fact, vectors are

technically defined by how they behave under transformations such as rotations and reflections. Since we really don't care about reflection properties at this point, we don't make a distinction between vectors and axial vectors.