Lesson 1 – Stationary Point Charges and Their Forces

1.0 Introduction

Every phenomenon related to electricity or magnetism, from static electricity to computer circuits to radio waves, depends directly on the force between point charges. But if we ask what charge is, we really don't have a good answer.

The ancient Greeks observed that amber, when rubbed, would attract pieces of straw. In fact, our word electricity comes from the Greek ἤλεκτρον (*élektron*) meaning amber. For centuries static electricity was studied as a curiosity, but it wasn't until the end of the 18th Century that scientists began serious study of electric and magnetic phenomena.

1.1 Basic Rules of Electrostatics

We now know that the source of charge on objects is either a net deficit or surplus of electrons. Because of their structure some materials readily give up electrons while others readily accept electrons. When charges are not moving, we call the charges on objects "static" electricity. The study of such charges is termed "electrostatics." The fundamental rules of electrostatics can be summarized as follows:

- "Positive" charge is produced on a glass rod by rubbing it with silk.

- "Negative" charge is produced on a rubber rod by rubbing it with fur.

- Charges move freely through certain materials, particularly metals. These are called "conductors." We know now that conductors differ from insulators in that atoms of a conductor have some electrons, the conduction electrons, that are not bound to individual atoms but are free to move throughout the solid.

- Charges remain fixed in place on other materials. These are called "insulators."

- Objects with like charge repel each other and objects with unlike charges attract each other.

- The force between charges is larger when the charges are closer together.

- If we denote the charge on an electron as $-e$ (note that e itself is positive), then all observed particles have charge $0, \pm e, \pm 2e,$ etc. We say that particles have "integral charge." (We believe, however, that the quarks that make up a proton come in charges of $+\frac{2}{3}e$ and $-\frac{1}{3}e$. Nevertheless, we have never observed any particle with such "fractional charge.") In this course, we will use SI (*Système Internationale*) units with very few exceptions. The SI unit of charge is the Coulomb (C). In SI units, $e = 1.602 \times 10^{-19}$ C.

- As far as we can tell, the total charge in the universe is a constant. (We can create matter out of energy or annihilate matter to create energy, but when we do so, the total charge of the particles we create or destroy must sum to zero.) This observation is called the "law of conservation of electric charge."

Things to remember:
- Charges are positive or negative. Likes charges repel. Unlike charges attract.
- Charge moves freely in conductors, but not in insulators.
- The force between charged particles is larger when the charges are closer together.
- Charge is quantized and conserved.

1.2 Our Understanding of Conductors

In the last section, we introduced the idea of conductors by giving you a simple working definition. You may have asked yourself why some materials are conductors and some materials are insulators. Scientists really didn't have answer to this question until quantum mechanics made its appearance in the early 20th century. To understand the details of the argument, you need to know some fairly sophisticated quantum mechanics. If you are willing to take a few things on faith, however, I can give you a simple explanation. First, we know that quantum mechanics predicts that electrons in an atom can occupy only certain energy levels or states. These states correspond to the electron orbitals you may have studied in chemistry classes. When we put a lot of atoms together to form a solid, neighboring atoms affect the electron states. In insulators, the net result is that all the electrons remain bound to nuclei. You can rip electrons off atoms or dump a few extra electrons onto atoms, but the electrons still remain bound in atoms. On the other hand, in conductors, the orbitals overlap so as to allow some of the outside electrons to be bound to the solid by electrostatic forces, but not to be bound to an individual nucleus. In this case electrons move around quite freely. In actuality, it does take a little energy to move electrons around in conductors, as they collide with other electrons and lose a little energy, but we will usually pretend that conductors are ideal and that it takes no energy at all to move electrons around.

While we're talking about conductors, there are two more types of conductors we ought to introduce: semiconductors and superconductors. We really won't pay much attention to either in this class, but it is useful to know about them.

Semiconductors are materials that have characteristics of both conductors and insulators. It their ground state, the electrons in a semiconductor are bound to individual atoms, as in an insulator. When the electrons are given some additional energy, however, they move into conduction states. By applying electric fields to a semiconductor, we can turn it from an insulator into a conductor at will. It is clear that such characteristics make semiconductors well-suited for digital applications.

Superconductors are materials which have no resistance at all. Through a strange quantum mechanical quirk, some materials have electrons with momenta that pair up in such a way that whenever one electron loses a certain amount of momentum to the solid, the other member of the pair gains precisely the same amount of momentum back from the solid. Once an electrical current is started in these materials, then it continues indefinitely, as long as you don't do any work with it like lighting a light bulb or turning a motor. Probably the most important use of superconductors is to produce very large currents for electromagnets. The downside to superconductors is that they need to be cooled. The most common superconducting material is lead. The temperature at which lead becomes superconducting, the critical temperature T_C, is 7.2 K, which is typically obtained by placing the lead in liquid helium. Some ceramic materials called high T_C superconductors have critical temperatures as high as 138 K. These materials are usually cooled to superconducting temperatures by using liquid nitrogen.

Things to remember:
- The electrons in conductors are not bound to individual atoms and are therefore free to move through the conductor with very little loss of energy.

1.3 Electrostatic Induction

We talked a little about forces between point charges, but we need to think a little about what happens with larger, everyday objects. Electrostatics was originally discovered when the ancient Greeks noticed that small pieces of straw were drawn to the surface of amber after it had been rubbed. Amber, like rubber, became negatively charged by rubbing. But the small pieces of straw were originally electrically neutral. Why would they be attracted to the amber? To understand this, we have to ask what happens to molecules in the straw. Straw is an insulator, so electrons are not free to move from atom to atom. However, if molecules have a positive end and a negative end, the molecules can rotate when amber is brought near, as in Fig. 1.1. The positive

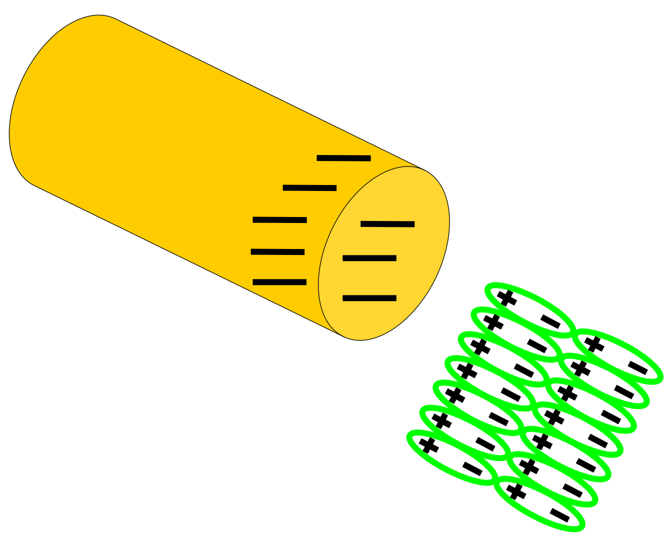


Figure1.1. Molecules in the straw aligning when amber is brought near.

ends of the molecules in the straw rotate toward the amber while the negative ends rotate away from the amber. The straw is still neutral; however, the positive charges in the straw are closer to the amber on average than the negative charges. The attractive force between the amber and the positive charges is therefore larger than the repulsive forces between the amber and the negative charges in the straw. So the straw is attracted to the amber.

If we replaced the straw with small bits of a conducting material, the electrons in the conductor would be repelled, leaving one side of the conductor positive and the other negative, in much the same way as the straw. The separation of positive and negative charges in a material is called "polarization." When such polarization occurs, there is an attractive force between the external charge and the polarized material.

This process, called electrostatic induction, is very common in our everyday experiences. When you rub balloons, they stick to neutral walls. When fabric picks up charge, it can cling to neutral body parts, and so on. In fact, it wasn't until 1620 that anyone reported observing electrostatic repulsion.

Think About It

If we rub a glass rod with silk and place it near (but not touching) a thin stream of water flowing from a faucet, the water is attracted to the rod. What does that tell us about the charge on the water?

If we rub a rubber rod with fur and do the same thing, the water is also attracted. Do you have to rethink your first answer? Can you explain why this happens?

It is helpful to know that water molecules are "polar." That means that although each molecule is neutral, one end is positively charged and the other is negatively charged.

Rubber soles on shoes are insulators, but if we rub rubber soles across a carpet, we can build up static electricity as evidenced by the spark that results when we touch a doorknob. Explain how this happens.

Things to remember:

- Electrostatic induction: When a charged object come closes to a second object, either a conductor or a polar insulator, charges within the second object are attracted or repelled. Since unlike charges are nearer each other, the net force is always attractive.

1.4 Fundamental Interactions, Virtual Particles, and Geometric Theories

The way we know that charges exist is by observing the forces that they produce. Just as with charge, forces are easy to describe, but understanding the origin of forces is quite another thing.

It may have been a few years since you took a mechanics course. In case that's true, let's review a few concepts about force. (If you feel that you need a more thorough review, you may want to look at a physics text for further details.) Our understanding of motion is based on Newton's three laws of motion:

1. The natural motion of an object is in a straight line at constant speed. That is, when the object isn't interacting with anything else in the universe, its velocity remains constant.
2. Momentum is defined as the product of mass and velocity. Force is defined as the rate momentum changes:

$$\vec{p} = m\vec{v}, \quad \vec{F} = \frac{d\vec{p}}{dt}$$

3. Forces result from the interaction between two objects. The force on object 1 from object 2 is equal in magnitude and opposite in direction to the force on object 2 from object 1. The net force on an object is the vector sum of the forces on the object resulting from interactions with all other objects in space.

You might find the statement of Newton's second law a little different than you remember. Note that for an object of constant mass, this equation reduces to the more familiar form

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

In the end, this means that we if we know the mass of an object and the object's position as a function of time, $\vec{r}(t)$, we can deduce:

1. the velocity of the object: $\vec{v} = \frac{d\vec{r}}{dt}$
2. the momentum of the object: $\vec{p} = m\vec{v}$
3. the acceleration of the object: $\vec{a} = \frac{d\vec{v}}{dt}$
4. the net force on the object: $\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$

Thus the force between point charges – or between any objects – is a quantity that can be determined experimentally. But, of course, physicists can't be content with measuring forces. We want to systematize, describe, and discuss the origin of forces as well. At a fundamental level, physicists generally recognize four forces, or interactions, between elementary particles: the gravitational force, the weak force, the electromagnetic force, and the strong force. The weak and strong forces are important only on subatomic scales. The strong force holds nuclei together and the weak force causes beta decay, among other things. In our everyday life, we deal principally with the gravitational and electromagnetic forces. In fact all the "usual" forces except gravity are, at the microscopic level, due to the electromagnetic force. These include contact forces, molecular forces, friction, spring forces, tension in ropes, etc.

Think About It

If two electrons are located near each other in space, we observe that they repel each other. How does one electron "know" that the second electron is present and that it should move away from it?

This is called the problem of "action at a distance." Physicists have two kinds of theories that explain action at a distance, but neither theory is wholly satisfactory.

The first type of theory is the "virtual particle" theory. We can think every real particle (electrons, protons, etc.) as constantly emitting a series of little balls, the "virtual particles." After each little ball is emitted, the electron that emits the ball recoils in order to conserve energy and momentum. If the little ball hits a second particle, it transfers energy and momentum to that particle.

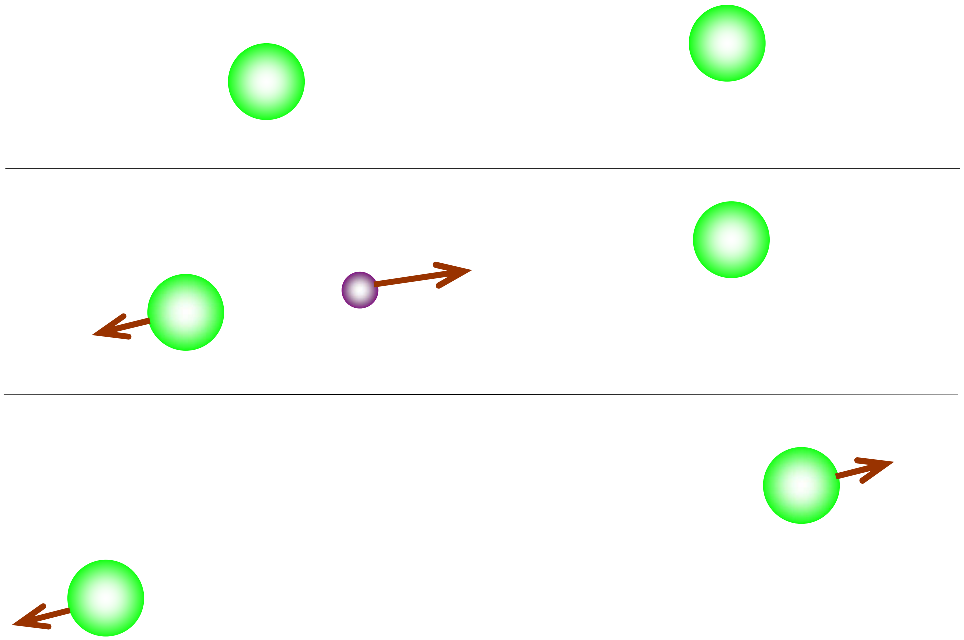


Figure 1.2. A "virtual particle" transferring force.

If we try to take this idea quite literally, we immediately come up with a host of objections:

- How does the real particle find the energy necessary to continuously produce virtual particles, and yet not lose energy of its own?
- If nothing absorbs the virtual particle, then does only the one real particle that emitted the virtual particle experience a force?
- How can virtual particles produce an attractive force?

In order to answer the first two objections, textbooks often describe virtual particle as being emitted and then reabsorbed by the same real particle unless they happen to strike another real particle. But we can then ask how it is that virtual particles can be induced to turn around and come back to where they started? The important point (that is sometimes overlooked in elementary texts) is that virtual particles are *virtual*. They are nothing more or less than convenient models used to describe terms in a mathematical series, and as such, the virtual particles can have unusual characteristics like imaginary mass.

Virtual particle theories work exceptionally well to describe the electromagnetic and weak interactions between elementary particles. The virtual particle theory of electromagnetic interactions is called Quantum Electrodynamics or QED. The virtual particles of QED are "virtual photons" as they have much in common with light. When the weak interaction is included under the umbrella of QED, the resulting theory is called Electroweak Theory. The virtual particles of the weak interaction are *W* and *Z* bosons – very massive particles that give rise to forces with extremely short ranges.

Physicists have tried to extend these same approaches to the strong and gravitational interactions, but with less success. The theory of the strong interaction requires us to think of protons, neutrons, and other similar particles to be composed of quarks. Quarks have electrical charge, but they also have another kind of charge that physicists call “color” as it comes in three different varieties. The virtual particle theory of the strong interaction is called Quantum Chromodynamics (QCD), but this theory is very complicated and difficult to test experimentally.

While details of QED are difficult, there are a few books that are quite good for general audiences. Two of these are *The Cosmic Onion* by Frank Close and *QED* by Richard Feynman.

Physicists have yet to produce a satisfactory virtual particle theory of gravity. Our theory of gravitation is Albert Einstein’s General Theory of Relativity. The basic idea of general relativity is that matter causes space-time (a four dimensional space that treats time as mathematically similar to a spatial dimension) to curve, and in turn curved space-time affects the motion of matter. How can we tell if space is curved? We believe that light goes in straight lines; however, if space curves, light rays would be expected to bend in order to follow the curvature of space. In 1915, Einstein predicted that light passing near the sun would be deflected by this effect. During a total eclipse of the sun in 1919, the apparent position of stars near the sun shifted, confirming that light did bend as it passed by the sun.

Some recommended textbooks about general relativity and gravitation are *Gravity from the Ground Up* by Bernard Schutz and *Exploring Black Holes: Introduction to General Relativity* by Edwin Taylor and John Wheeler. Another book that is more suitable for an advanced undergraduate is *Gravity: An Introduction to Einstein’s General Theory of Relativity* by James Hartle.

Einstein spent much of the latter part of his life working for a way to bring these two very different approaches into a single framework. However, he was never able to accomplish it. While we continue to make progress in our efforts to understand force at a fundamental level, we still have no satisfactory unified theory of all the forces.

Now, you are probably asking yourself what you need to know about all this. Don’t be concerned about all the details, but you should be able to summarize in a few sentences the basic ideas behind virtual particle theories (QED) and geometric theories (General Relativity).

Things to remember:

- Forces can be thought to arise from either the exchange of virtual particles (as with QED) or from the modification of the curvature of space-time (as with general relativity).

1.5 Physical Models

If you ask an introductory physics student what physics is, you’ll often get the response that physics is learning a lot of equations so you can plug numbers in and get answers out. Hopefully by this point you have learned that physics isn’t just equations. It requires understanding how nature works in terms of ideas and concepts as well.

To see how ideas and equations relate to each other, let's consider light as an example. (These examples may be familiar to you, but if not, you needn't be concerned about it.) As we study light in physics classes, we usually think of it either as a wave or as a particle. If we think of light as a wave, we can describe interference phenomena, such as the patterns produced by light passing through a pair of narrow slits. Describing light as a wave, we can draw pictures of how the waves from the slits add with each other when light emerges at different angles. (If you want to see the details, just look up "double slit interference" in any standard physics text.) On the basis of the model, we can come up with the equation

$$m\lambda = d \sin\theta \, .$$

In order to use the equation you need to know what each of the variables means and how they relate to quantities that you will see in the problems we give you to solve. We won't bother explaining that here, as that's not the point of this discussion. The point is that we thought of light as a wave, visualized wavefronts geometrically as if light were a wave on a string or waves in a lake, and made a prediction about how the light should behave when it passes through two narrow slits. In other words, we used a "wave model" of light to predict its behavior. The equation in the end is only as good as the model and, moreover, it can be understood only if we understand the model that created it. It is true that if I were to assign a problem where I gave you m, λ, and d, you could give me a value for θ. However, this would require no understanding of the physics and no more than high school-level mathematical skill. To understand the physics, you really need to understand the model.

On the other hand, we can think of light as made of massless particles called "photons." We can think of each photon as carrying momentum and energy. When a photon interacts with an electron, it can undergo a process called Compton scattering. In this process, we think of the photon as a little ball that transfers energy and momentum to the electron in a collision much like the collision of billiard balls. In this case, we get the equation:

$$\Delta\lambda = \frac{h}{m_0 c}\left(1 - \cos\theta\right).$$

Again, we won't bother describing what each symbol means. The point here is that, to describe Compton scattering we can use a "particle model" of light. Again we can arrive at an equation, but we really need to understand the model before we can make sense of the equation.

So which model is true? Is light a wave, or is it a particle. We often say that it is both. What we mean by this is that in some applications light behaves as it were a wave. In other applications it behaves as if it were composed of particles. In reality, light is neither wave nor particle, it is just light. Neither model is completely true; both models are useful in the range of experiments where they be applied. The models we use in physics are not necessarily descriptions of the truth. They are symbolic systems with mathematical symbols that behave in limited ways like the physical quantities themselves. – Or, if you don't like the philosophical-sounding words: science can't say a lot about truth – it can only produce models that behave a lot like real physical systems.

In this lesson we introduce two different, although related, models to describe electromagnetic phenomena. You will first learn the "thread model," which explains the interaction of point charges. This model helps us understand why there is magnetism, why electromagnetic forces behave the way they do, why there is radiation, and why we can induce electric currents to generate power. In short, the thread model answers a lot of important "why" questions.

The second model you will learn is the "field model." This is traditional model that grew out of the 19th Century concepts of electric and magnetic field lines. It is naturally better-suited to describing forces caused by distributions of charges and currents. The field model leads mathematically to Maxwell's Equations, the differential equations that form the basis of most serious electromagnetic calculations.

But we need to remember that neither model is really true. You might be asking, "Why don't you just teach us the truth?" There are two reasons for that. What we think of as the closest approximation to the truth, QED, is so complicated that it is completely impractical for most problems. The second reason is that even QED has its philosophical problems and shouldn't be thought of as the truth either.

Things to remember:

- Physical models are analogies that behave much like systems that can be found in nature. Good models behave very much like physical systems over a lot of applications. Physical models are not necessarily "true."

1.6 The Thread Model in Electrostatics

To this point we have philosophized quite a bit and given a few rules about the qualitative behavior of the force between charged particles. What we want to do is to be able to write down some mathematical relationships that we can use to make quantitative predictions. To do this, I am going to propose a model of the electromagnetic interaction. In the spirit of the previous section, I make no claim that the model is "truth." In fact, I know that it has certain deficiencies when I consider the behavior of very small systems that require quantum mechanical explanations. But the model does describe all the characteristics of electric and magnetic interactions in systems that are large enough for quantum mechanical effects to average out. We call this model the "Thread Model" of the electromagnetic interaction.

Electric and magnetic phenomena are all caused by the charge-related forces between elementary particles. To begin with, let us consider one elementary particle that we choose to call, somewhat arbitrarily the "source particle." The source particle exerts a force on a second particle that we call the "field particle." We will also assume the field particle is at rest. (Of course we could call the second particle the source particle and the first particle the field particle if we were interested in calculating the force on the first particle.) According to the Thread Model, the source particle emits threads which cause the force felt by the field particle. The basic rules of these threads are:

1. A thread is line segment emitted by a source particle. One end is called a "head" and the other end called a "tail." In Figs. 1.3 and 1.4, the head is the solid circle and the tail is the open circle. The head and tail both come off from the source in the same direction. That is, if the head goes off along the *y* axis, the tail also goes off along the *y* axis. (We will find in Chapter 10 that this is not true if the source particle is accelerating, however.)
2. The thread (head and tail) moves in a straight line at the speed of light.

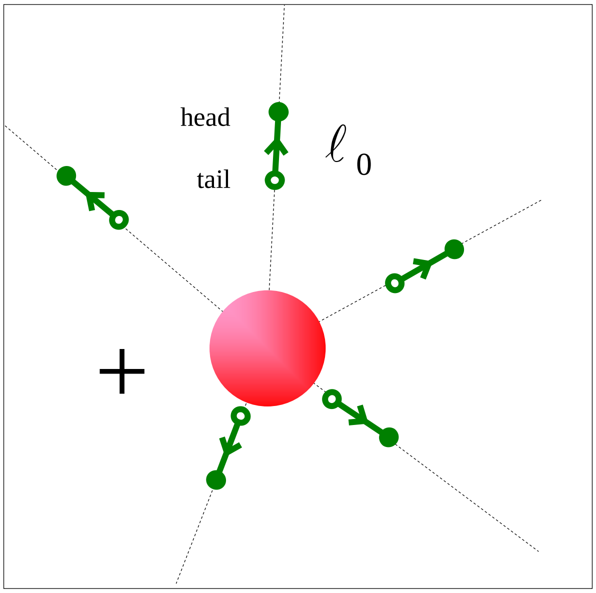


Figure 1.3. Threads of a positive source charge.

3. Threads are emitted in random directions, with the total number of threads emitted per second, N_0, very large. The length of each thread, ℓ_0, is very small. The product of N_0 and ℓ_0 must have units of *m/s*, and hence is a velocity. *For an electron*, we let the product $N_0\ell_0 = c$ where *c* is the speed of light.
4. The number of threads emitted per second, N_0, is proportional to the charge of the source particle. That is, if the source has twice the charge, it emits twice as many threads.
5. Each thread has a direction. The direction of threads emitted by positive source particles is from the tail toward the head. The direction of threads emitted from negative source charges is from the head toward the tail. In these figures, the arrow in the middle of the thread denotes its direction.

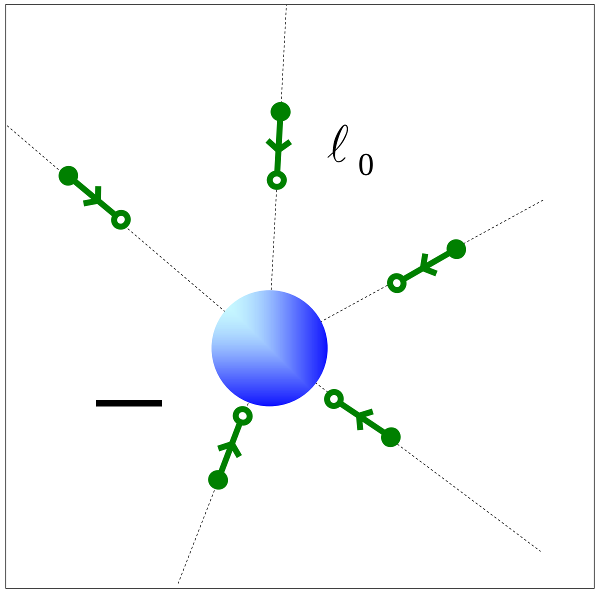


Figure 1.4. Threads of a negative source charge.

6. The direction of the force felt by a field particle is determined by the direction of *the threads that surround the field particle*. The force is in the same direction as the threads' arrows if the field particle is positive and in the opposite direction if the field particle is negative. Thus, like charges repel and unlike charges attract.

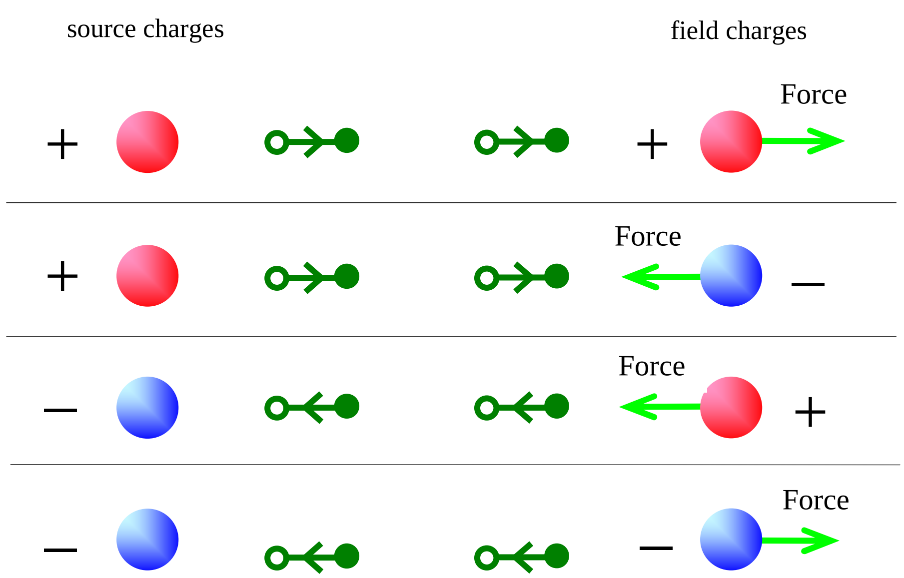


Figure 1.5. The direction of forces between charges.

7. The magnitude of the force felt by the field particle is proportional to the product of the charge of the field particle, the length of the threads near the field particle, and the density (number per unit volume) of the threads surrounding the field particle. Threads do

not carry any energy or momentum. They produce a force by modifying space near the field particle. Threads are not modified as they pass through matter.

To get a better feeling for the threads of a point charge, see Fig. 1.6, which shows many threads in three dimensions. For simplicity, I have omitted the arrows from the threads. But threads are not stationary, so to better appreciate how they behave, we need an animated version of this figure. One such animation is available on the course website:

http://www.physics.byu.edu/faculty/rees/220/AVIfiles/StatThreads.avi

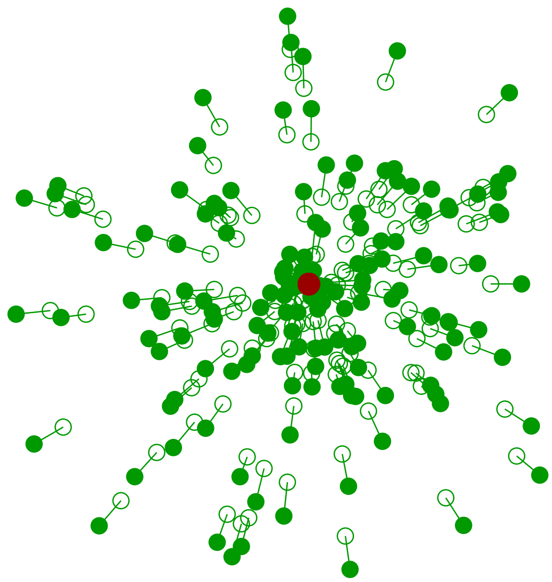


Figure 1.6. The threads of a point charge.

From what you have learned about the Thread Model, you should be able to answer the following three questions:

1) If we double the charge of the source particle, what happens to the force on a field particle?
2) If the source particle is positive and the field particle is negative, what is the direction of the force of a field particle?
3) If a field particle is moved to twice its original distance from a point source particle, what happens to the force?

Now, here are the answers; but think about the questions before you read on!

1) Since the number of threads doubles, the force doubles.

2) The direction of the force is radially inward; the field particle will be attracted toward the source particle.
3) The force is smaller by a factor of four. The same number of threads found at the smaller radius has to spread out over a sphere of twice the radius, so the density of threads drops by a factor of four. (Remember that the surface area of a sphere is $A = 4\pi r^2$, so the area of a sphere is four times larger when the radius doubles.)

Things to remember:
• A simplified version of the rules for the threads of a stationary charge:

1. The head and tail both of a thread come off from the source in the same direction.
2. The thread moves in a straight line at the speed of light.
3. Many short threads are emitted in random directions.
4. The number of threads emitted is proportional to the charge of the source.
5. For a positive charge, the direction of the thread is away from the source, for a negative charge, it is toward the source.
6. The direction of the force on a field particle is in the direction of the threads surrounding the field particle if the field particle is positive and opposite the direction of the threads if the field particle is negative.
7. The force on a field particle is proportional to a) the charge of the field particle, b) the density of threads near the field particle, c) the length of the threads near the field particle.

1.7 Obtaining Coulomb's Law from the Thread Model

We can qualitatively use the Thread Model to give us a mental picture of what happens when charges exert forces on each other – that's a very useful thing, so be sure you can visualize the threads and how they behave. However, we can also use this model to find an equation for the force between stationary charges. Let's put our source particle at the origin of a coordinate system and a field particle at a distance *r* from the origin in some arbitrary direction. We call the point where the source is located *S* and the point where the field particle is located *P* (as *F* would be confused with the force.) We call the vector from the origin to the field particle $\vec{r}_0$.

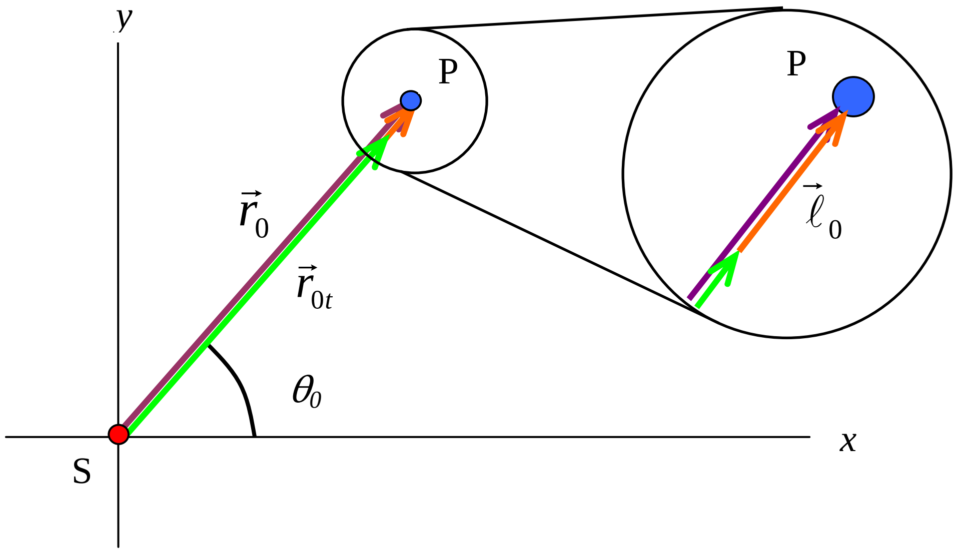


Figure 1.7. Defining the vector $\vec{r}_0$. If the head of a thread is at $\vec{r}_0$, then its tail is at $\vec{r}_{0t}$.

We can combine Rules 6 and 7 of the previous section into an equation that describes the thread force:

(1.1 Thread force) $$\vec{F} = \frac{e}{\varepsilon_0} q_f \vec{\ell}_0 \nu_0$$

where:

$\vec{F}$ is the force on the test charge in newtons (N), a vector.
e is the charge of an electron in coulombs (C) .
ε_0 is a constant that we will determine later.
q_f is the charge of the field particle in coulombs (C).
$\vec{\ell}_0$ is the length of a thread in meters (m), a vector.
ν_0 is the density of threads in threads per cubic meter (number / m^3 , or m^{-3}).

Now we will use Eq. (1.1) to find a simple equation for the force between two stationary point charges. We assume that we know q_f and ℓ_0. We also know that for a positive source at rest, the threads point radially outward. We can write a unit vector in this direction as $\hat{r}_0$. Let's review a few things about unit vectors that you may have forgotten. The vector from the source to the point P is $\vec{r}_0$. This magnitude of this vector is written r_0. The unit vector $\hat{r}_0$ is a vector pointing in the direction of $\vec{r}_0$ with a magnitude of 1. It is clear that $\frac{\vec{r}_0}{r_0}$ points in the direction of

$\vec{r}_0$ and its magnitude is $|\hat{r}_0| = \frac{|\vec{r}_0|}{r_0} = \frac{r_0}{r_0} = 1$, so $\hat{r}_0 = \frac{\vec{r}_0}{r_0}$. In terms of the unit vector, we have $\vec{\ell}_0 = \ell_0 \hat{r}_0$.

We also need to find the density of threads ν_0 near point *P*. Because of the spherical symmetry of the problem, no direction in space is special for any reason at all. That means that the density of threads at a distance r_0 form the source in one direction must be the same as the density of threads as a distance r_0 in any other direction. however, since the threads are spreading out, they become less dense as r_0 increases. So let's consider a spherical shell of thickness dr_0 located a distance r_0 from the source.

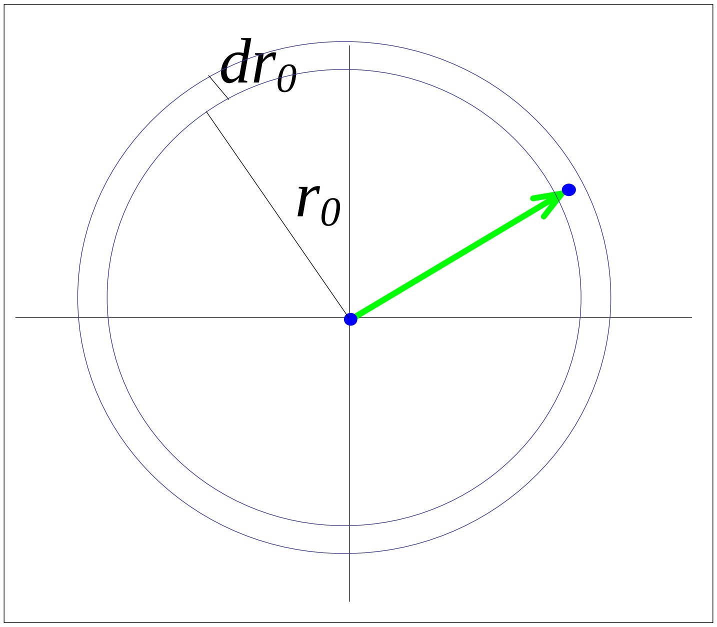


Figure 1.8. The shell we use to calculate the density of threads at r_0.

(Note that when we write a quantity as differential, such as dr_0, we assume that dr_0 is very small and that we will eventually take the limit $dr_0 \to 0$ whether we explicitly state it or not.)

The density of threads in this shell is just

$$\nu_0 = \frac{number\ of\ threads\ in\ the\ shell}{volume\ of\ the\ shell}.$$

Let's start by finding the volume of the shell. I'm sure you've done this in calculus classes before, but we'll review the concept a little. *Note that we will make use of this result frequently throughout the course, so it's good to pay careful attention!* Think of the shell as the peel of an orange. Peel the orange and smash the peel flat on a table. The volume of peel itself (not including the gaps between the pieces of the peel) is just the surface area of the orange multiplied by the thickness of the peel. You might object that the surface area of the orange isn't quite exactly the same on the outside of the peel and on the inside of the peel, but we'll choose to ignore this fact as we can make the "peel" of our spherical shell as thin as we want. That is, we let $dr_0 \to 0$. Since the surface are of a sphere is just $4\pi r_0^2$, the volume is $4\pi r_0^2\, dr_0$.

Now we need to find the number of threads in the shell. Let us assume that we know the number of threads emitted by an electron (or proton) in one second. We'll call this quantity N_{0e}. Since the number of threads emitted by the source particle is proportional to its charge, we know that the total number of threads it emits in one second must be $N_{0e}\frac{q_s}{e}$ where q_s is the charge of the source in coulombs and e is the magnitude of the electron charge in coulombs. This quantity is the total number of threads emitted in one second, but the threads that are in the shell were emitted over a much smaller time than one second. To find the number of threads in the shell, we need to know the difference between the time when the threads at the outside the shell were emitted and the time when the threads at the inside of the shell were emitted. To obtain the number of threads in the shell, we multiply the number of threads emitted per second times this time interval. To find the time interval, we know that the threads travel at the speed of light c and that the thickness of the shell is dr_0. Since $c = \frac{dr_0}{dt}$, we can deduce that $dt = \frac{dr_0}{c}$.

Finally, we can put this all together:

$$\vec{F} = \frac{e}{\varepsilon_0} q_f \vec{\ell}_0 \nu_0 = \frac{e}{\varepsilon_0} q_f \ell_0 \hat{r}_0 \frac{1}{4\pi r_0^2\, dr_0} \frac{N_{0e} q_s}{e} \frac{dr_0}{c}$$

Using the relation we gave earlier, $N_{0e}\ell_0 = c$, we can simplify this to:

(1.2 Coulomb's Law) $$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_s q_f}{r_0^{\,2}} \hat{r}_0$$

where:

- $\vec{F}$ is the force on the test charge in newtons (N), a vector.
- ε_0 is a constant called the "permittivity of free space." Its value is $8.85\times10^{-12}\ C^2/N\,m^2$.
- q_s is the charge of the source particle in coulombs (C).
- q_f is the charge of the field particle in coulombs (C).
- $\vec{r}_0$ is the vector from the source particle to the field particle. It is in meters (m).

The constant ε_0 is something we cannot predict from our model. We have to measure the force between charges and see what value of ε_0 gives us the correct force. Also notice that we assumed both charges were positive when we obtained Coulomb's law above. We need to think about what happens when one or both of the charges are negative. Fortunately, a minus sign in the vector equation just reverses the direction of the force, giving us the correct result without having to change the formula. (If that isn't obvious, take a minute to convince yourself that Eq. (1.2) is true for all combinations of positive and negative charge!) Lastly, we chose to put our source particle at the origin of a coordinate system. Note that the formula doesn't require this as long we choose to interpret $\vec{r}_0$ as the vector from the source particle to the field particle.

Things to remember:
- You do not need to reproduce the mathematics that led us to Coulomb's law.
- Know Coulomb's law $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_s q_f}{{r_0}^2}\hat{r}_0$ and be able to use it.

1.8 Using Coulomb's Law

Coulomb's law was named after Charles Augustin de Coulomb (1736-1806) who derived the law in 1785 from experimental observations. Coulomb's law was rather remarkable in that it almost perfectly mirrored Newton's law of gravitation

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}\,.$$

Coulomb's law and Newton's law of gravitation do differ in two important respects, however. 1) Coulomb's law allows for repulsive as well as attractive forces since charge can be either positive or negative, and 2) the constant $(1/4\pi\varepsilon_0)$ that appears in Coulomb's equation equals $8.99\times10^{+9}$ in SI units, whereas G equals 6.67×10^{-11} in SI units. What this means is that in a hydrogen atom, for example, the Coulomb force is approximately 2×10^{39} times larger than the gravitational force!

When you actually use Coulomb's law to calculate the force between stationary point charges, the one thing that sometimes is difficult is the unit vector. Remember that this is a unit vector along the line from the source particle to the field particle. As we mentioned above, you can find this unit vector by using the relation $\hat{r}_0 = \vec{r}_0 / r_0$. However, in practice it is better to multiply both the numerator and the denominator of Eq (1.2) by r_0 to obtain a form that is easier for computation:

(1.3 Coulomb's law, alternate form) $$\vec{F} = \frac{q_s q_f \vec{r}_0}{4\pi\varepsilon_0 r_0^3}\,.$$

Hence, if we know the vector from the source particle to the field particle, either in terms of magnitude and direction or in terms of components, we can easily find the force on the field charge. Just be careful to remember that Coulomb's law is really an inverse square law, not an inverse cube law! It's really inverse square because there is a hidden factor of r_0 in the numerator of Eq. (1.3).

Example 1.1. The Coulomb Force between a Proton and an Electron

A proton is located at the origin of a coordinate system. An electron is located at the point 125 nm $\hat{x}$ -67.0 nm $\hat{y}$.

First, we see that the vector $\vec{r}_0$ from the source to the electron is just the quantity that is given:

$$\vec{r}_0 = 125 \text{ nm } \hat{x} \text{ -67.0 nm } \hat{y}.$$

Once we have $\vec{r}$ we can then obtain r_0 by Pythagoras's relationship:

$$r_0 = x\hat{x} + y\hat{y} = \sqrt{x^2 + y^2} = 142 \text{ nm}$$

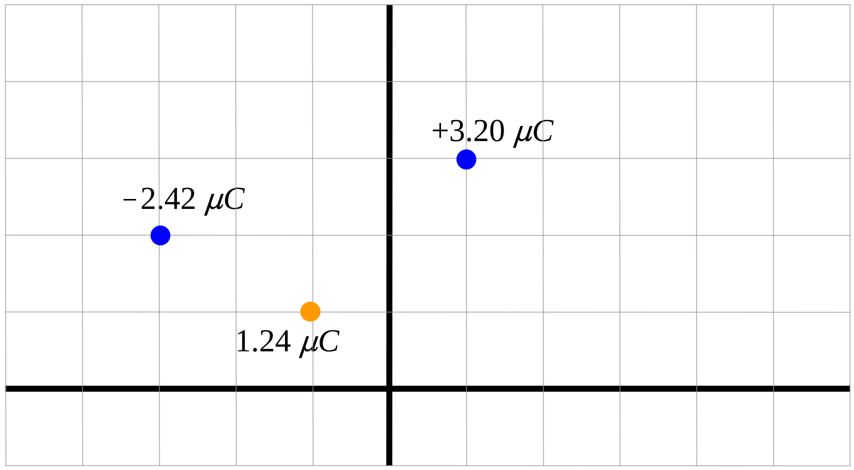
Now, we can just plug numbers into Coulomb's law. there is just one caution. Electromagnetic units can be very difficult to relate to each other. While it goes against what you were told in other physics courses, it is usually advisable to ignore units in this course. In order to avoid problems; however, it is important to put *everything* in SI units. If you do this, your answer will always be in SI units as well.

$$\vec{F} = \frac{q_s q_f \vec{r}_0}{4\pi\varepsilon_0 r_0^3} = \frac{-\left(1.60\times10^{-19}\right)^2\left(125\times10^{-9}\hat{x} - 67.0\times10^{-9}\hat{y}\right)}{4\pi\times8.85\times10^{-12}\left(142\times10^{-9}\right)^3}N = \left(10.1\times10^{-15}\hat{x} - 5.41\times10^{-15}\hat{y}\right)N$$

It becomes a more challenging problem when we have the source at some point other than the origin or if we have two different sources in a problem. Let's look at an example where both of these complications come into play:

Example 1.2. The Coulomb Force of Multiple Charges

A 3.20 μC charge is located at the point (1.00 *cm*, 3.00 *cm*) and a – 2.42 μC charge is located at the point (–3.00 *cm*, 2.00 *cm*). Find the force on a 1.24 μC charge located at the point (-1.00 *cm*, 1.00 *cm*).



First, draw the force vectors on the diagram. You'll need to look at the signs of the charges to determine the direction of the forces. Roughly draw the vectors to scale. Which vector do you think is bigger? (It's hard to be sure without actually calculating numbers, isn't it!) Add your force vectors together as arrows.

The problem is essentially the same as the previous problem, except that we need to calculate the force from each of the two charges and then add them together as vectors. Let's begin by finding the vector *from* each source charge *to* the field charge. We'll call the vector on the left 1 and the vector on the right 2. We'll convert all distances to *m* so that everything is in SI units.

$$\vec{r}_1 = 0.02\hat{x} - 0.01\hat{y}$$
$$r_1 = \sqrt{0.02^2 + 0.01^2}$$
$$\vec{r}_2 = -0.02\hat{x} - 0.02\hat{y}$$
$$r_2 = \sqrt{0.02^2 + 0.02^2}$$

Now all we need to do is plug the known values into Coulomb's law:

$$\vec{F}_1 = \frac{q_1 q_f \vec{r}_1}{4\pi\varepsilon_0 r_1^3} = \frac{(-2.42\times10^{-6})(1.24\times10^{-6})(0.02\hat{x} - 0.01\hat{y})}{4\pi\times8.85\times10^{-12}\left(\sqrt{0.0005}\right)^3} N = -48.3N\,\hat{x} + 24.1N\,\hat{y}$$
$$\vec{F}_2 = \frac{q_2 q_f \vec{r}_2}{4\pi\varepsilon_0 r_2^3} = \frac{(3.20\times10^{-6})(1.24\times10^{-6})(-0.02\hat{x} - 0.02\hat{y})}{4\pi\times8.85\times10^{-12}\left(\sqrt{0.0008}\right)^3} N = -31.5N\,\hat{x} - 31.5N\,\hat{y}$$
$$\vec{F} = \vec{F}_1 + \vec{F}_2 = -79.8N\,\hat{x} - 7.4N\,\hat{y}$$

Now, for the fun of it, we can calculate the magnitudes of each of the forces.

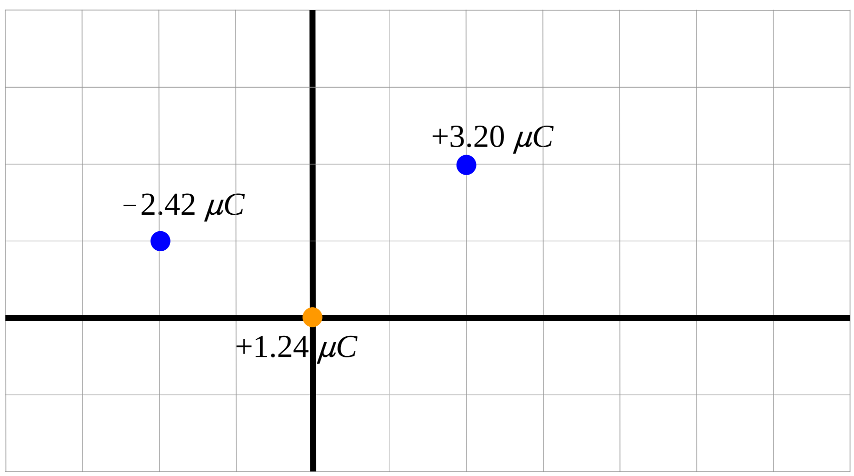
$$F_1 = \sqrt{48.3^2 + 24.1^2}\,N = 54.0\,N$$
$$F_2 = \sqrt{31.5^2 + 31.5^2}\,N = 44.6\,N$$
$$F = \sqrt{79.8^2 + 7.4^2}\,N = 80.1\,N$$

Were your arrows roughly correct?

What angle does the force make with respect to the *x* axis?

$$\tan\theta = \frac{F_y}{F_x} = \frac{-7.4\,N}{-79.8\,N}$$
$$\theta = -175°$$

Finally, note that the force can't be affected by our choice of the origin of the coordinate system. (Why not?) So, we can redraw the picture with the field charge at the origin. This shouldn't affect our calculation above. Can you explain why it doesn't make any difference to the numerical answer?



Things to remember:

• Memorize and be able to use the form $\vec{F} = \dfrac{q_s q_f \vec{r}_0}{4\pi\varepsilon_0 r_0^3}$

• Be sure to review how to add vectors and to obtain magnitudes and directions!!

1.9 Force, Energy, and Work

Newton's second law is really a definition of force: $\vec{F}_{net} = \frac{d\vec{p}}{dt}$. Thus, forces cause changes in the momentum of objects. You know that when you apply a force to an object, you can also change its kinetic energy. If you drop a ball from rest, the ball's kinetic energy increases. If you catch a ball someone throws to you, you apply a force that decreases its kinetic energy. If you swing a rock on the end of a string, the string applies a force that keeps the rock moving in a circular path, but does little to change the energy of the rock. You studied how all these process worked in your mechanics course; however, it is probably useful to review the concepts of energy and work. For more details, you should refer to any standard physics text.

Let's consider an object that moves a very short distance $d\ell$ while a force is being applied to it, as shown in Fig. 1.9. Since the object moves a very short distance, it is safe to consider the path to be along a straight line and the force to be constant.

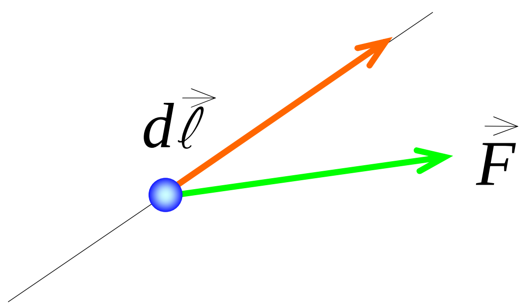


Figure 1.9. An object moves along the path $d\vec{\ell}$ under the influence of a force $\vec{F}$.

Let us begin by calculating the rate at which the kinetic energy changes in this process. We'll need to break the velocity into components and apply the chain rule as we take derivatives of each component. You'll also need to remember that the dot product of two vectors $\vec{A}$ and $\vec{B}$ is $\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$. (If you've forgotten this, it might be a good to look over the vector section of Review now.)

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}m\frac{d}{dt}\left(v_x^2 + v_y^2 + v_z^2\right) = \frac{1}{2}m\left(2v_x\frac{dv_x}{dt} + 2v_y\frac{dv_y}{dt} + 2v_z\frac{dv_z}{dt}\right)$$

$$\frac{dK}{dt} = m\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\frac{d\vec{p}}{dt} = \vec{v}\cdot\vec{F}$$

The rate at which kinetic energy changes is called the "mechanical power" of a system. It is just equal to the dot product of force and velocity. This tells us that the part of the force that is perpendicular to the velocity can not change the kinetic energy. Similarly, the part of the force parallel (or antiparallel) to the velocity increases (or decreases) the kinetic energy. If we translate that into normal English sentences, it tells us: When you push an object in the direction it's moving, it moves faster. If you push an object opposite the direction it's moving, it goes slower. If you push an object sideways (with respect to its direction of motion), it changes direction without speeding up or slowing down. If you think about that, it should make intuitive sense.

Furthermore, we know that in time *dt*, the object moves a distance $d\vec{\ell}$, so the velocity is just $\vec{v} = \frac{d\vec{\ell}}{dt}$. Thus, the change in kinetic energy of the object along this very small path is

$$\frac{dK}{dt} = \vec{F} \cdot \frac{d\vec{\ell}}{dt}$$
$$\Rightarrow dK = \vec{F} \cdot d\vec{\ell}.$$

Now, if an object moves along some path between points A and B, we can find the total change of kinetic energy by adding up all the contributions from each tiny line segment, as schematically illustrated in Fig. 1.10.

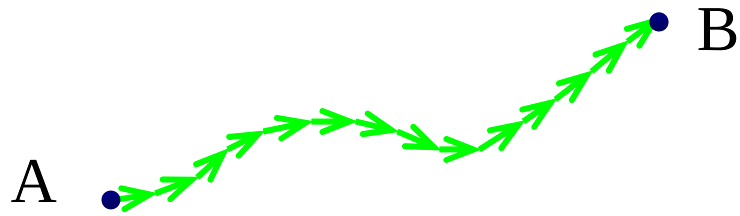


Figure 1.10. The path from A to B can decomposed into many small steps. Each step has a different $d\vec{\ell}$ and may have a different $\vec{F}$.

To find the change in kinetic energy along the entire path, we simply add up all the *dK* from each tiny step. That is:

(1.6 Work-Energy Theorem) $\Delta K_{A\rightarrow B} = \int_{A\rightarrow B} dK = \int_{A\rightarrow B} \vec{F} \cdot d\vec{\ell} = W$

This result, known as the Work-Energy Theorem, simply states that the work done in going from A to B along a specified path is equal to the change in kinetic energy along that path.

This kind of integral is called a line integral. It looks complicated, but don't be frightened by it. You won't actually have to evaluate it except along a couple of simple paths. The way we evaluate it in hard problems is to do exactly what we just outlined (except we let a computer do the arithmetic). We approximate the path by a series of straight line segments $d\vec{\ell}$, find the force at each segment, take the dot product of $d\vec{\ell}$ and the force on each segment, and add all these together.

For a certain restricted class of forces, it turns out that the work done in going from a point A back to the same point A is always zero. That means that if we start at point A with a certain kinetic energy, whenever we come back to the point A, no net work has been done to the object, and its kinetic energy remains the same. If this happens to be the case, we can also say something about the work done in going from A to B along two different paths.

First, however, we need to ask ourselves how the work in going from A to B relates the work in going from B to A. Think of a ball being tossed upward and then falling to the point from which it started. The amount of kinetic energy lost by the ball as it goes up is the same as the amount gained by the ball coming down. That is $\Delta K_{up} = -\Delta K_{down}$. We can generalize this by noting that when we do the line integral from B to A, the integral is just the same as that from A to B, except that every $d\vec{\ell}$ is reversed in direction, as shown in Fig. 1.11,

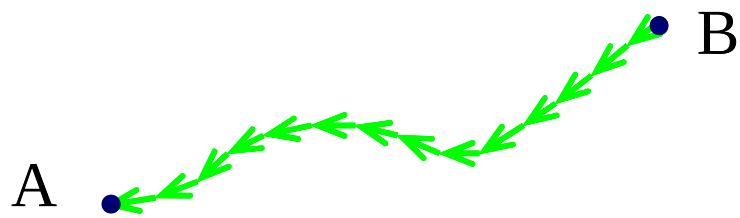


Figure 1.11. When we do the line integral from B to A, every $d\vec{\ell}$ is reversed.

That means that very generally

$$\int_{A\to B} \vec{F}\cdot d\vec{\ell} = -\int_{B\to A} \vec{F}\cdot d\vec{\ell}\,.$$

Now let's compare the work or change in kinetic energy when we go along two different paths from point A to point B, where we assume the line integral from A back to A is zero.

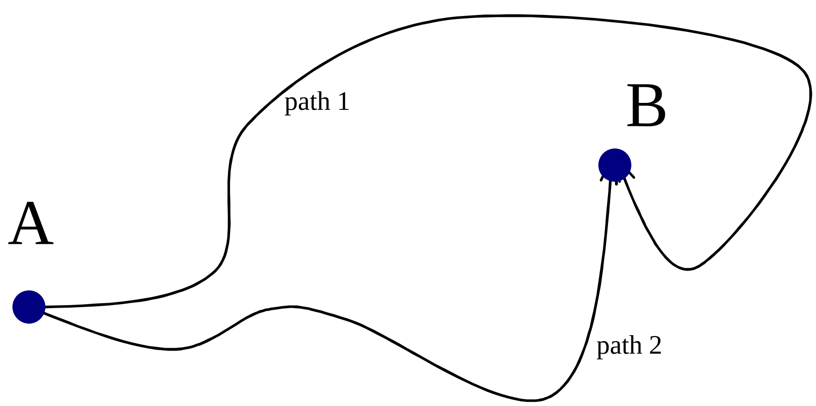


Figure 1.12. Calculating the work in going from A to B along two different paths.

$$\int_{A\to B,\,path\,1} \vec{F}\cdot d\vec{\ell} + \int_{B\to A,\,path\,2} \vec{F}\cdot d\vec{\ell} = 0$$
$$\int_{A\to B,\,path\,1} \vec{F}\cdot d\vec{\ell} - \int_{A\to B,\,path\,2} \vec{F}\cdot d\vec{\ell} = 0$$
$$\int_{A\to B,\,path\,1} \vec{F}\cdot d\vec{\ell} = \int_{A\to B,\,path\,2} \vec{F}\cdot d\vec{\ell}$$

The bottom line to this review is: As long as the work in going around any closed path is zero, the work in going from point any A to any point B is independent of the path we choose. That, in turn, means that if we know the kinetic energy of an object at A, we know what its kinetic energy will be at B, no matter what path the object took in going to B.

Although we could continue to talk about work and kinetic energy, it is easiest at this point to introduce the concept of potential energy. Let's restrict ourselves to cases where the force is in one Cartesian direction (x,y,z) or in the radial direction (r) in either cylindrical or spherical coordinates. Potential energy is then defined as:

(1.7 Potential energy from force) $U(x) = -\int F dx, \quad U(r) = -\int F dr$

Note that this is essentially the same as work except that the sign is reversed and the integral is an indefinite integral rather than an integral along a specific path. Evidently, the work done in going from A to B is:

$$W_{A\to B} = \int_{x_A}^{x_B} F dx = -U(x_B) + U(x_A) = -\Delta U_{A\to B} = \Delta K_{A\to B}$$
$$\Delta K_{A\to B} + \Delta U_{A\to B} = 0$$

Since the change in kinetic energy is equal in magnitude and opposite in sign to the change in potential energy, we can define a total energy E that remains the same at any point. That is:

(1.8 Conservation of energy) $K + U = E$, E is constant.

Other useful relationships can be obtained by taking the derivatives of Eq. (1.7), so we can find the force when the potential energy is given.

(1.9 Force from potential energy) $F = -\frac{dU}{dx}, \quad F = -\frac{dU}{dr}$

If you have taken a course in vector calculus, then you will appreciate the following general relationship between force and potential energy:

$$\vec{F} = -\nabla U \equiv -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

The triangle is called the "gradient" and it is defined in terms of the partial derivatives as written. If you don't know anything about gradients, don't worry about it. We'll learn a little about them later in the course. All of our problems in this chapter will be restricted to one dimension.

Things to remember:

• Potential energy and force are related by the equations $U(x) = -\int F dx, \quad F = -\frac{dU}{dx}$.

• When potential energy can be defined, $U + K = E$ which is constant.

1.10 Using Potential Energy

Position, time, mass, momentum, and kinetic energy are all things we can measure in nature. Potential energy, on the other hand is not. We can only infer a value for the potential energy by either assuming energy conservation or integrating the force. For example, hold a pencil over the floor. What is its potential energy? You might say it's *mgy* because you remember the formula from mechanics. While that's true, what does *y* mean? Do you measure *y* from the floor, the top of a table, the ground outside? It doesn't really matter, because the only time we can "see" potential energy is when it changes, and the change in potential energy is the same, no matter where we choose to call $y = 0$. Mathematically, this is reflected by the fact that potential energy is defined as an indefinite integral, so we can always add an arbitrary constant of integration to it. In electromagnetic theory, we have conventions as to where we usually call potential energy zero, but remember that these will just be conventions.

When we derived our expression for potential energy, we stipulated that the results were only valid if $\oint_{A\to A} \vec{F} \cdot d\vec{\ell} = 0$. (The circle on the integral sign means that we evaluate the integral around a closed path.) Of course, this isn't always true. In particular, it is usually **not** true when any of the following conditions are met:

- the force changes in time (such as the force from moving magnets, changing currents in circuits, etc.).
- the force depends on the direction of motion (such as drag forces and frictional forces).
- the force tends to move objects around circular paths.

For electricity and magnetism, the only time we can define potential energy is when the forces are caused by stationary charges. We cannot define potential energy for electric forces from moving charges or for any magnetic forces. (In fact, all magnetic forces are produced by moving charges).

Example 1.5. The Potential Energy of a Constant Force

An object of mass 1.2 kg feels a constant force of $F = 6.00$ N in the $\hat{z}$ direction.

(a) Find a general expression for the potential energy as a function of *z*.

$$U(z) = -\int F dz = -F\int dz = -Fz + C \quad \text{where } C \text{ is a constant of integration.}$$

(b) If we want U(0) = 0, then what is the value of the constant of integration?

$$U(0) = 0 + C = 0 \Rightarrow C = 0$$

(c) If the object is released from rest at *z = 0* and then moves to *z = 5 m*, what is its kinetic energy?

$$\begin{aligned}
&K + U = E \\
&at\ z = 0\ we\ know\ 0 + 0 = E \\
&at\ z = 5\ we\ know\ K + (-6N \times 5m) = E = 0 \\
&\Rightarrow K = 30\ J
\end{aligned}$$

(d) How does this answer differ if we choose to let U(0) = 5 *J* ?

$$\begin{aligned}
&K + U = E \\
&at\ z = 0\ we\ know\ 0 + 5\ J = E \\
&at\ z = 5\ we\ know\ K + (-6N \times 5m + 5\ J) = E = 5\ J \\
&\Rightarrow K = 30\ J
\end{aligned}$$

Example 1.6. Potential Energy and the Coulomb Force

(a) Using Coulomb's law, Eq. (1.2), find the potential energy associated with this force.

$$\begin{aligned}
&\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_s q_f}{r_0^{\ 2}}\hat{r}_0 \\
&U(r) = -\int F(r)dr = -\int \frac{1}{4\pi\varepsilon_0}\frac{q_s q_f}{r_0^{\ 2}}dr = -\frac{1}{4\pi\varepsilon_0}q_s q_f \int r^{-2} dr \\
&U(r) = +\frac{1}{4\pi\varepsilon_0}\frac{q_s q_f}{r} + C
\end{aligned}$$

(b) The conventional choice for the constant of integration is to let U go to zero when charges are separated by a very large distance. What is the value of *C*?

$$0 = \lim_{r\to\infty} U(r) = \lim_{r\to\infty} \frac{1}{4\pi\varepsilon_0}\frac{q_s q_f}{r} + C = 0 + C$$

$$\Rightarrow C = 0$$

Since this is a very useful result, we'll put it in a box:

(1.7 The potential energy of two point charges) $U(r) = +\frac{1}{4\pi\varepsilon_0}\frac{q_s q_f}{r}$

Note that if you have three point charges, the potential energy of the entire system is the sum of the potential energies of the **three pairs** of charges added together.

(c) All objects tend to move toward smaller potential energy. If both the source charge and the field charge are positive, where is the potential energy the smallest? What if one charge is positive and one is negative? Is this consistent with what we know about when the Coulomb force is attractive and when it is repulsive?

If both charges are positive, the potential energy is also positive everywhere. It is a minimum when *r* goes to infinity. This indicates that the force is repulsive. If one charge is positive and the other is negative, the potential energy is negative everywhere. It is a minimum when *r* goes to zero. This indicates that the force is attractive.

Things to remember:

- The potential energy of two point charges is $U(r) = +\frac{1}{4\pi\varepsilon_0}\frac{q_s q_f}{r}$.
- Potential energy can be defined for any number of charges, as long as they are stationary