

Appendix C – Details of Thread Theory

C.0 Introduction

In the text offered a few quantitative details about thread theory, but I omitted many of the details as they aren't really necessary to understand and use theory at the level I expect in this course. We often do that in introductory texts and recommend that students refer to more advanced texts for the details. There are, however, no published sources for further details about thread theory, so I felt that I should provide them in an appendix. You should be able to follow the mathematics of this appendix with the background you have. The algebra can get involved, but the concepts are not particularly challenging. If you have not yet read Appendix A, however, you will find that it will be helpful in understanding the Lorentz transformations that we will use.

C.1. Two Useful Mathematical Rules

There are two important rules involving dot and cross products that we will apply regularly in this section. These are the “dot-cross rule” and the “BAC minus CAB rule.” You may have encountered these rules in math courses before.

The dot-cross rule is

$$(C.1 \text{ The dot-cross rule}) \quad \vec{A} \cdot (\vec{A} \times \vec{B}) = 0.$$

We know that $\vec{A} \times \vec{B}$ is perpendicular to \vec{A} , and that the dot product of perpendicular vectors is zero. Formally, we may write:

$$\begin{aligned} \vec{A} \cdot (\vec{A} \times \vec{B}) &= \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{bmatrix} \\ &= A_x(A_y B_z - A_z B_y) + A_y(A_z B_x - A_x B_z) + A_z(A_x B_y - A_y B_x) \\ &= A_x(A_y B_z - A_z B_y - A_y B_z + A_z B_y) + A_y A_z B_x - A_z A_y B_x = 0 \end{aligned}$$

Also, it is clear that $\vec{B} \cdot (\vec{A} \times \vec{B}) = 0$ as well. In fact any linear combination of \vec{A} and \vec{B} dotted into $\vec{A} \times \vec{B}$ must give 0 as well:

$$(C.2 \text{ The generalized dot-cross rule}) \quad (a\vec{A} + b\vec{B}) \cdot (\vec{A} \times \vec{B}) = 0.$$

The BAC minus CAB rule doesn't have as simple a geometrical interpretation as the dot-cross rule, but is very useful in simplifying vector expressions.

$$(C.3 \text{ The BAC minus CAB rule}) \quad \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

The proof is done by brute force:

$$\text{Let } \vec{D} \equiv \vec{B} \times \vec{C} = \begin{bmatrix} B_y C_z - B_z C_y \\ B_z C_x - B_x C_z \\ B_x C_y - B_y C_x \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{A} \times \vec{D} = \begin{bmatrix} A_y D_z - A_z D_y \\ A_z D_x - A_x D_z \\ A_x D_y - A_y D_x \end{bmatrix} = \begin{bmatrix} A_y (B_x C_y - B_y C_x) - A_z (B_z C_x - B_x C_z) \\ A_z (B_y C_z - B_z C_y) - A_x (B_x C_y - B_y C_x) \\ A_x (B_z C_x - B_x C_z) - A_y (B_y C_z - B_z C_y) \end{bmatrix}$$

$$= \begin{bmatrix} B_x (A_y C_y + A_z C_z) - C_x (A_y B_y + A_z B_z) \\ B_y (A_z C_z + A_x C_x) - C_y (A_z B_z + A_x B_x) \\ B_z (A_x C_x + A_y C_y) - C_z (A_x B_x + A_y B_y) \end{bmatrix} = \begin{bmatrix} B_x (A_y C_y + A_z C_z + A_x C_x) - C_x (A_y B_y + A_z B_z + A_x B_x) \\ B_y (A_z C_z + A_x C_x + A_y C_y) - C_y (A_z B_z + A_x B_x + A_y B_y) \\ B_z (A_x C_x + A_y C_y + A_z C_z) - C_z (A_x B_x + A_y B_y + A_z B_z) \end{bmatrix}$$

$$= \begin{bmatrix} B_x (\vec{A} \cdot \vec{C}) - C_x (\vec{A} \cdot \vec{B}) \\ B_y (\vec{A} \cdot \vec{C}) - C_y (\vec{A} \cdot \vec{B}) \\ B_z (\vec{A} \cdot \vec{C}) - C_z (\vec{A} \cdot \vec{B}) \end{bmatrix} = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$$

C.2 Relations Involving Angles and Vector Lengths

We will assume that the source is moving with a velocity $\vec{\beta}$ in a reference frame that we call the “lab frame” or the \mathbf{R} frame. In this frame, the points S , T , U , and P , are defined as in Lesson 2. We reproduce Figs. 2.1 and 2.4 here for convenience.

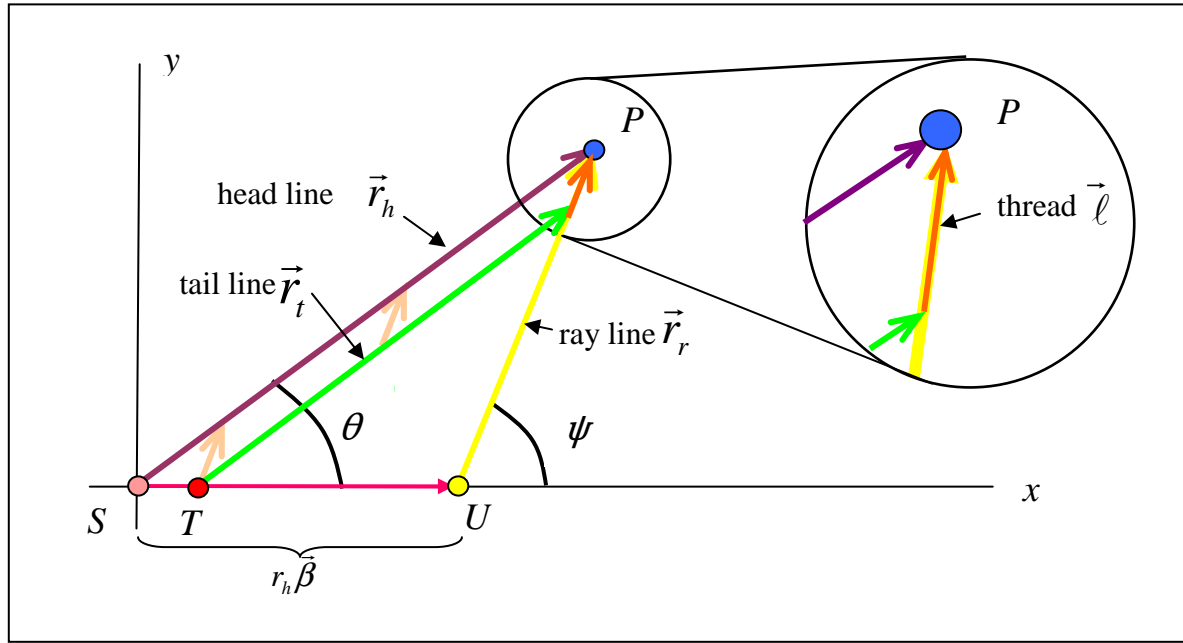


Figure C.1. The geometry of the lab frame (\mathbf{R}).

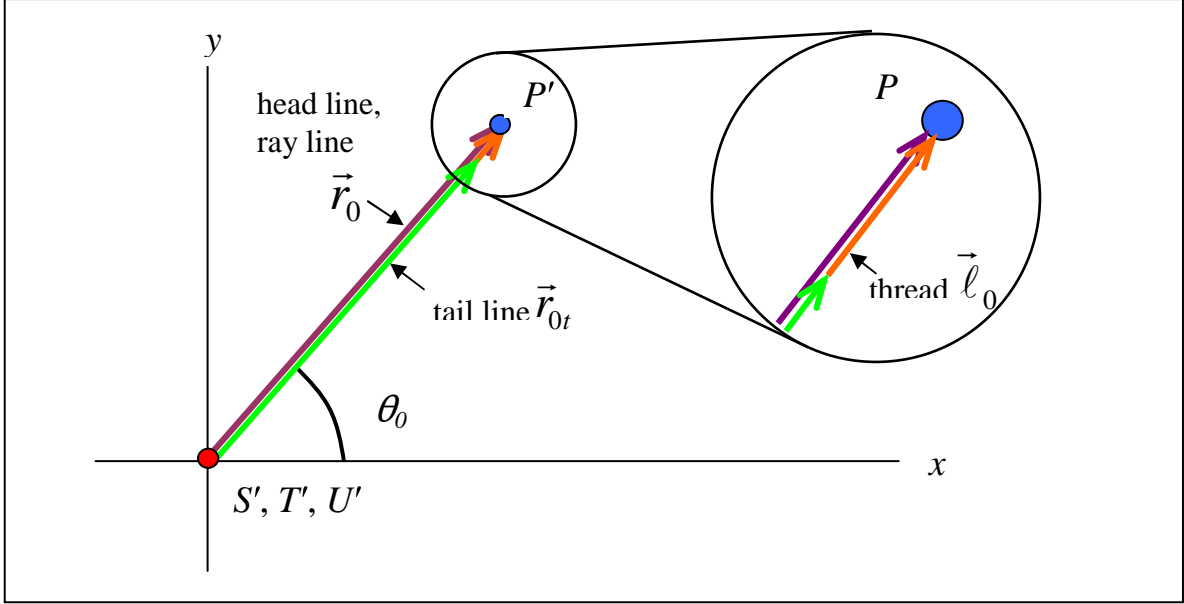


Figure C.2 The geometry in the rest frame of the source (\mathbf{R}').

To relate the various vectors, let's find the events (space and time) that correspond to S (source when the head is emitted), T (source when the tail is emitted), U (source when the threads arrives at P), and P (thread arriving at a field point) in both reference frames, \mathbf{R} and \mathbf{R}' . We will use primes or 0 subscripts for quantities in the rest frame. We'll denote four-vectors for the events with the obvious (I hope) notation: $\mathbf{r}_s, \mathbf{r}'_U$, etc.

$$\begin{aligned} \mathbf{r}'_s &= \begin{bmatrix} 0 \\ \vec{0} \end{bmatrix} & \mathbf{r}_s &= \begin{bmatrix} 0 \\ \vec{0} \end{bmatrix} \\ \mathbf{r}'_T &= \ell_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \mathbf{r}_T &= \gamma \ell_0 \begin{bmatrix} 1 \\ \vec{\beta} \end{bmatrix} \\ \mathbf{r}'_U &= r_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \mathbf{r}_U &= r_h \begin{bmatrix} 1 \\ \vec{\beta} \end{bmatrix} \\ \mathbf{r}'_P &= r_0 \begin{bmatrix} 1 \\ \hat{u}' \end{bmatrix} & \mathbf{r}_P &= r_h \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} \end{aligned}$$

Let me explain why we know each of these four-vectors.

In the rest frame \mathbf{R}' , the source remains at the origin, so events S' , T' , and U' all have spatial components $\vec{0}$. Recall that the time component of a four-vector is actually ct , which represents the distance light – or a thread – travels in time t . Thus ct' for event T' is the distance the head travels during the time the thread is emitted. This is just the thread length ℓ_0 . We know the distance from the origin to point P' is r_0 and we call \hat{u}' the direction from the origin to P' , hence we know \mathbf{r}'_P . We call the direction of the thread's motion \hat{u} and \hat{u}' just to emphasize the

fact that threads are massless particles that travel at the speed of light and hence transform as described in Sect. B.6. At times we call the same directions \hat{r}_h and \hat{r}_0 to emphasize their connection with the head lines in the two reference frames. Finally, the event U' corresponds to the source at the time \mathbf{R}' when the head arrives at P' , so $ct' = r_0$.

In the lab frame (the frame where the source is in motion) \mathbf{R} , the source is also at the origin at the time $ct=0$ when the head is emitted. Due to dilation of time, it takes longer, $ct = \gamma \ell_0$, for the thread to be emitted. During the time the thread is being emitted, the source moves to the location $\vec{v}t = \vec{\beta}ct = \vec{\beta}\gamma\ell_0$.

The thread eventually moves a distance r_h to point P , taking time $ct = r_h$. The direction the thread travels to point P is called \hat{u} . And when the thread arrives at P , the source arrives at U with time $ct = r_h$ and position $r_h\vec{\beta}$.

Events S , T , and P are the same events viewed in the two reference frames, so the corresponding four-vectors are related by the Lorentz transformation. U and U' , on the other hand, are two distinct events. Recall that two events that are simultaneous in one frame are not generally simultaneous in another. Since U is defined by being simultaneous with P , we cannot expect the same event to be simultaneous with P' . However, for S , T , and P , we can write:

$$\begin{aligned} \mathbf{r}'_S &= \mathbf{L}(\vec{\beta})\mathbf{r}_S & \mathbf{r}'_T &= \mathbf{L}(\vec{\beta})\mathbf{r}_T & \mathbf{r}'_P &= \mathbf{L}(\vec{\beta})\mathbf{r}_P \\ \mathbf{r}_S &= \mathbf{L}(-\vec{\beta})\mathbf{r}'_S & \mathbf{r}_T &= \mathbf{L}(-\vec{\beta})\mathbf{r}'_T & \mathbf{r}_P &= \mathbf{L}(-\vec{\beta})\mathbf{r}'_P \end{aligned}$$

Let's first apply the Lorentz transformation to event P .

$$(C.4) \quad \begin{aligned} \mathbf{r}'_P &= \mathbf{L}(\vec{\beta})\mathbf{r}_P & r_0 \begin{bmatrix} 1 \\ \hat{u}' \end{bmatrix} &= \mathbf{L}(\vec{\beta})r_h \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = r_h \begin{bmatrix} \gamma(1 - \vec{\beta} \cdot \hat{u}) \\ \hat{u} - \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u}) \end{bmatrix} \\ \mathbf{r}_P &= \mathbf{L}(-\vec{\beta})\mathbf{r}'_P & r_h \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} &= \mathbf{L}(-\vec{\beta})r_0 \begin{bmatrix} 1 \\ \hat{u}' \end{bmatrix} = r_0 \begin{bmatrix} \gamma(1 + \vec{\beta} \cdot \hat{u}') \\ \hat{u}' + \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u}') \end{bmatrix} \end{aligned}$$

The unit vectors are a little awkward, so let's express the dot products in terms of angles:

$$\begin{aligned} \vec{\beta} \cdot \hat{u} &= \beta \cos \theta \\ \vec{\beta} \cdot \hat{u}' &= \beta \cos \theta_0 \end{aligned}$$

We can obtain two convenient relationships for the r_h and r_0 by equating the time (top) components of each of these vector equations and substituting for the dot products:

$$(C.5) \quad \begin{aligned} r_0 &= r_h \gamma(1 - \beta \cos \theta) \\ r_h &= r_0 \gamma(1 + \beta \cos \theta_0) \end{aligned}$$

Two more useful relationships can be obtained by taking the dot product of the spatial parts of Eq. (C.4) with $\vec{\beta}$:

$$\begin{aligned}
r_0 \beta \cos \theta_0 &= r_h (\beta \cos \theta - \gamma \beta^2 + \kappa \beta^3 \cos \theta) \\
&= r_h [\beta \cos \theta (1 + \kappa \beta^2) - \gamma \beta^2] \\
&= r_h [\gamma \beta \cos \theta - \gamma \beta^2] \text{ as } \kappa = (\gamma - 1) / \beta^2 \\
&= r_h \beta \gamma (\cos \theta - \beta) \\
r_h \beta \cos \theta &= r_0 (\beta \cos \theta_0 + \gamma \beta^2 + \kappa \beta^3 \cos \theta_0) \\
&= r_0 [\beta \cos \theta_0 (1 + \kappa \beta^2) + \gamma \beta^2] \\
&= r_h [\gamma \beta \cos \theta_0 + \gamma \beta^2] \\
&= r_0 \beta \gamma (\cos \theta_0 + \beta)
\end{aligned}$$

Dividing out the common factors of β and substituting for r_h / r_0 from Eq. (C.5), we have:

$$\begin{aligned}
\cos \theta_0 &= \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \\
\cos \theta &= \frac{\cos \theta_0 + \beta}{1 + \beta \cos \theta_0}
\end{aligned}$$

Now let's see how the ray line transforms. (Note that the sum and difference of four-vectors are also four-vectors to which we can apply the Lorentz transformation.) The ray line transformed into the rest frame is a four-vector with a spatial component that is just \vec{r}_0 but with a time component that is the difference in ct between the events P and U as viewed in the rest frame.

$$\begin{aligned}
\mathbf{r}_P - \mathbf{r}_U &= r_h \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} - r_h \begin{bmatrix} 1 \\ \vec{\beta} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{r}_r \end{bmatrix} \\
\mathbf{L}(\vec{\beta})(\mathbf{r}_P - \mathbf{r}_U) &= \begin{bmatrix} \gamma & -\gamma \vec{\beta}^T \\ -\gamma \vec{\beta} & \mathbf{I} + \kappa \vec{\beta} \vec{\beta}^T \end{bmatrix} \begin{bmatrix} 0 \\ \vec{r}_r \end{bmatrix} = \begin{bmatrix} -\gamma \vec{\beta} \cdot \vec{r}_r \\ \vec{r}_r + \kappa \vec{\beta} (\vec{\beta} \cdot \vec{r}_r) \end{bmatrix} = \begin{bmatrix} ct' \\ r_0 \hat{u}' \end{bmatrix}
\end{aligned}$$

The spatial part of this gives:

$$\vec{r}_0 = r_0 \hat{u}' = \vec{r}_r + \kappa \vec{\beta} (\vec{\beta} \cdot \vec{r}_r)$$

First, we take the dot product of this with $\vec{\beta}$ to get a relation involving the angle ψ :

$$\begin{aligned}
r_0 \hat{u}' \cdot \vec{\beta} &= \vec{r}_r \cdot \vec{\beta} + \kappa \beta^2 (\vec{\beta} \cdot \vec{r}_r) \\
r_0 \beta \cos \theta_0 &= \beta r_r \cos \psi (1 + \kappa \beta^2) = \gamma \beta r_r \cos \psi \\
r_0 \cos \theta_0 &= \gamma r_r \cos \psi
\end{aligned}$$

A similar relation comes from the magnitude of the cross product:

$$\begin{aligned}
|r_0 \hat{u}' \times \vec{\beta}| &= |\vec{r}_r \times \vec{\beta} + \kappa (\vec{\beta} \times \vec{\beta})(\vec{\beta} \cdot \vec{r}_r)| \\
r_0 |\hat{u}' \times \vec{\beta}| &= |\vec{r}_r \times \vec{\beta}| \text{ as } (\vec{\beta} \times \vec{\beta}) = 0 \\
r_0 \beta \sin \theta_0 &= r_r \beta \sin \psi \\
r_0 \sin \theta_0 &= r_r \sin \psi
\end{aligned}$$

Combining these last two results, we can also write:

$$\tan \psi = \gamma \tan \theta_0.$$

From Fig. C.1, we can also write down some geometrical relationships that relate the angles ψ and θ . Recall that ψ is the angle between $\vec{\beta}$ and \vec{r}_r .

$$\begin{aligned}
r_h \sin \theta &= r_r \sin \psi \\
r_h \cos \theta &= r_h \beta + r_r \sin \psi
\end{aligned}$$

We can combine these relationships to obtain some additional useful equations:

$$\begin{aligned}
r_r^2 \sin^2 \psi + r_r^2 \cos^2 \psi &= r_r^2 = (r_h \sin \theta)^2 + (r_h \cos \theta - r_h \beta)^2 \\
r_r^2 &= r_h^2 (1 + \beta^2 - 2\beta \cos \theta) \\
r_r &= r_h \sqrt{1 + \beta^2 - 2\beta \cos \theta} \\
\frac{r_r \sin \psi}{r_r \cos \psi} &= \tan \psi = \frac{r_h \sin \theta}{r_h \cos \theta - r_h \beta} \\
\tan \psi &= \frac{\sin \theta}{\cos \theta - \beta}
\end{aligned}$$

We can summarize these results as follows:

$$(C.6) \quad \frac{r_h}{r_0} = \frac{1}{\gamma(1 - \beta \cos \theta)} = \gamma(1 + \beta \cos \theta_0)$$

$$(C.7) \quad \cos \theta_0 = \frac{\cos \theta - \beta}{1 - \beta \cos \theta}, \quad \cos \theta = \frac{\cos \theta_0 + \beta}{1 + \beta \cos \theta_0}$$

$$(C.8) \quad r_h \sin \theta = r_r \sin \psi = r_0 \sin \theta_0$$

$$(C.9) \quad r_r = r_h \sqrt{1 + \beta^2 - 2\beta \cos \theta}$$

$$(C.10) \quad \tan \psi = \frac{\sin \theta}{\cos \theta - \beta} = \lambda \tan \theta_0$$

C.3 Moving Source Charge, Stationary Field Charge

When a source charge is moving at constant velocity, the head and tail are emitted at the same angle in the lab frame. In the rest frame, the head and tail are also emitted at the same angle, though the angle is not the same as the angle in the frame of the moving source. Let's go back to the rest frame and write the space-time four-vectors for the head and tail of such a pair. We'll call the times at which we measure the head and tail positions ct'_h and ct'_t , respectively. We need to keep in mind that in the rest frame the tail is emitted at time $ct'_t = \ell_0$.

$$\mathbf{r}'_h = ct'_h \begin{bmatrix} 1 \\ \hat{u}' \end{bmatrix} \quad \mathbf{r}'_t = \begin{bmatrix} \ell_0 \\ 0 \end{bmatrix} + (ct'_t - \ell_0) \begin{bmatrix} 1 \\ \hat{u}' \end{bmatrix}.$$

The first term in the last equations tells us that the tail is emitted from the origin at time $ct'_t = \ell_0$. The second term tells us that once emitted, the tail travels in the same direction as the head for a time $ct'_t - \ell_0$. Transforming these into the lab frame gives:

$$\begin{aligned} \mathbf{r}_h &= \mathbf{L}(-\vec{\beta}) \mathbf{r}'_h = ct'_h \begin{bmatrix} \gamma(1 + \vec{\beta} \cdot \hat{u}') \\ \hat{u}' + \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u}') \end{bmatrix} \\ \mathbf{r}_t &= \mathbf{L}(-\vec{\beta}) \mathbf{r}'_t = \begin{bmatrix} \gamma\ell_0 \\ \gamma\ell_0\vec{\beta} \end{bmatrix} + (ct'_t - \ell_0) \begin{bmatrix} \gamma(1 + \vec{\beta} \cdot \hat{u}') \\ \hat{u}' + \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u}') \end{bmatrix} \end{aligned}$$

We can subtract the two four-vectors to get the thread vector. Since we want the head and tail to be measured at the same time, the time component of this difference must be zero.

$$\begin{aligned} \mathbf{r}_h - \mathbf{r}_t &= (ct'_h - ct'_t + \ell_0) \begin{bmatrix} \gamma(1 + \vec{\beta} \cdot \hat{u}') \\ \hat{u}' + \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u}') \end{bmatrix} - \gamma\ell_0 \begin{bmatrix} 1 \\ \vec{\beta} \end{bmatrix} \\ & \quad (ct'_h - ct'_t + \ell_0)\gamma(1 + \vec{\beta} \cdot \hat{u}') - \gamma\ell_0 = 0 \\ \mathbf{r}_h - \mathbf{r}_t &= \frac{\ell_0}{1 + \vec{\beta} \cdot \hat{u}'} \begin{bmatrix} \gamma(1 + \vec{\beta} \cdot \hat{u}') \\ \hat{u}' + \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u}') \end{bmatrix} - \gamma\ell_0 \begin{bmatrix} 1 \\ \vec{\beta} \end{bmatrix} \\ \vec{\ell} &= \frac{\hat{u}' + \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u}')}{1 + \vec{\beta} \cdot \hat{u}'} \ell_0 - \gamma\ell_0\vec{\beta} = \gamma\ell_0\hat{u} - \gamma\ell_0\vec{\beta} \quad \text{by Eq. (B.9)} \\ (C.11) \quad \vec{\ell} &= \gamma\ell_0(\hat{u} - \vec{\beta}) = \gamma\ell_0 \frac{\vec{r}_r}{r_h} \quad \text{since } \vec{r}_r = \vec{r}_h - r_h\vec{\beta} = r_h(\hat{u} - \vec{\beta}) \end{aligned}$$

This is the same result we found in Eq. (2.3) or Eq. (A.10).

C.4 Moving Field Charge, Stationary Source Charge

By determining the lengths and densities of threads at a field point, we can calculate the force on a field particle as long as the field particle is at rest. To account for the motion of the field particle we introduced the concept of stubs, but did so without proving why stubs work. In this section, we will use the Lorentz transformation on forces to show the need for stubs and the origin of the Lorentz force.

Let us take a very general force produced by a moving source charge on a stationary field particle. We will call the lab frame \mathbf{R}_1 and the rest frame of the source charge \mathbf{R}_0 . We will add subscripts 0 and 1 to about everything in sight so that we are clear which quantity belongs to which to frame. We assume that the source charge is at the origin of the lab coordinate system at time $ct=0$ when a thread is emitted. The velocity of the source charge is $\vec{\beta}_{s1}$ in this frame. The thread reaches a field charge at point P a short time later. The field charge moves with velocity $\vec{\beta}_{f1}$

In the lab frame, we can write the space-time four vectors for various events as follows:

Point S as the thread leaves the source

$$\mathbf{r}_{S1} = \begin{bmatrix} 0 \\ \vec{0} \end{bmatrix}$$

Point P as the thread arrives

$$\mathbf{r}_{P1} = \begin{bmatrix} ct_1 \\ \vec{r}_{h1} \end{bmatrix} = \begin{bmatrix} r_{h1} \\ \vec{r}_{h1} \end{bmatrix} = r_{h1} \begin{bmatrix} 1 \\ \hat{u}_1 \end{bmatrix}$$

Note that $r_{h1} = ct$ because the thread travels at the speed of light.

Point U , the source when the thread arrives at P

$$\mathbf{r}_{U1} = r_{h1} \begin{bmatrix} 1 \\ \vec{\beta}_{s1} \end{bmatrix}$$

In addition, we can write the energy-momentum four vectors for the two charges:

Source charge, rest mass = E_{s0}

$$\mathbf{e}_{s1} = \begin{bmatrix} E_{s1} \\ \vec{p}_{s1}c \end{bmatrix} = E_{s0}\gamma_{s1} \begin{bmatrix} 1 \\ \vec{\beta}_{s1} \end{bmatrix}$$

Field charge, rest mass = E_{f0}

$$\mathbf{e}_{f1} = \begin{bmatrix} E_{f1} \\ \vec{p}_{f1}c \end{bmatrix} = E_{f0} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Points S and P define the head line, of course. We can also define the ray line \vec{r}_r and the projection of the ray line on the head line, ρ , as follows:

$$\begin{aligned} \vec{r}_{r1} &= \vec{r}_{h1} - r_{h1}\vec{\beta}_{s1} = r_{h1}(\hat{u}_1 - \vec{\beta}_{s1}) \\ \rho_1 &= \hat{u}_1 \cdot \vec{r}_{r1} = r_{h1}(1 - \hat{u}_1 \cdot \vec{\beta}_{s1}) \end{aligned}$$

In this reference frame, we also know that the force is given by Eqs. (2.6) and (A.8):

$$\vec{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{\vec{r}_r}{\gamma_{s1}^2 \rho_1^3}.$$

The force four-vector (see Sect. B.2) on the field particle is:

$$\mathbf{f}_1 = \gamma_{f1} \begin{bmatrix} \vec{F}_1 \cdot \vec{\beta}_{f1} \\ \vec{F}_1 \end{bmatrix} = \frac{1}{E_{0f}} \begin{bmatrix} \vec{F}_1 \cdot \vec{p}_{f1} c \\ \vec{F}_1 E_{f1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{F}_1 \end{bmatrix} \text{ as the field particle is at rest.}$$

The first thing we want to do is find the force on the field charge in the rest frame of the source. This is an interesting question in itself, but also is the easiest way to find the general force on a moving field charge. We begin by transforming everything to the rest frame of the source.

$$\begin{aligned} \mathbf{r}_{S0} &= \mathbf{L}(\vec{\beta}_{s1}) \mathbf{r}_{S1} = \begin{bmatrix} \gamma_{s1} & -\gamma_{s1} \vec{\beta}_{s1}^T \\ -\gamma_{s1} \vec{\beta}_{s1} & \mathbf{I} + \kappa \vec{\beta}_{s1} \vec{\beta}_{s1}^T \end{bmatrix} \begin{bmatrix} 0 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} 0 \\ \vec{0} \end{bmatrix} \\ \mathbf{r}_{P0} &= \mathbf{L}(\vec{\beta}_{s1}) \mathbf{r}_{P1} = r_{h1} \begin{bmatrix} \gamma_{s1} & -\gamma_{s1} \vec{\beta}_{s1}^T \\ -\gamma_{s1} \vec{\beta}_{s1} & \mathbf{I} + \kappa \vec{\beta}_{s1} \vec{\beta}_{s1}^T \end{bmatrix} \begin{bmatrix} 1 \\ \hat{u}_1 \end{bmatrix} = r_{h1} \begin{bmatrix} \gamma_{s1} (1 - \vec{\beta}_{s1} \cdot \hat{u}_1) \\ \hat{u}_1 - \gamma_{s1} \vec{\beta}_{s1} + \kappa \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \hat{u}_1) \end{bmatrix} = r_0 \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix} \end{aligned}$$

The event corresponding to the location of the source at the time the thread arrives at point P is

$$\mathbf{r}_{U0} = \begin{bmatrix} r_0 \\ \vec{0} \end{bmatrix}.$$

Note that this is not the same event as U in the lab frame, as events that are simultaneous in the lab frame are not generally simultaneous in the rest frame.

$$\begin{aligned} \mathbf{e}_{s0} &= \mathbf{L}(\vec{\beta}_{s1}) \mathbf{e}_{s1} = E_{s0} \gamma_{s1} \begin{bmatrix} \gamma_{s1} & -\gamma_{s1} \vec{\beta}_{s1}^T \\ -\gamma_{s1} \vec{\beta}_{s1} & \mathbf{I} + \kappa_{s1} \vec{\beta}_{s1} \vec{\beta}_{s1}^T \end{bmatrix} \begin{bmatrix} 1 \\ \vec{\beta}_{s1} \end{bmatrix} = E_{s0} \gamma_{s1} \begin{bmatrix} \gamma_{s1} (1 - \beta_{s1}^2) \\ \vec{\beta}_{s1} - \gamma_{s1} \vec{\beta}_{s1} + \kappa_{s1} \vec{\beta}_{s1} (\beta_{s1}^2) \end{bmatrix} \\ &= E_{s0} \gamma_{s1} \begin{bmatrix} \gamma_{s1} / \gamma_{s1}^2 \\ \vec{\beta}_{s1} - \gamma_{s1} \vec{\beta}_{s1} + (\gamma_{s1} - 1) \vec{\beta}_{s1} \end{bmatrix} \text{ as } \kappa \beta^2 = \gamma - 1 \\ &= E_{s0} \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} \end{aligned}$$

This result shouldn't be too surprising, as we're transforming to the rest frame of the source particle.

$$\mathbf{e}_{f0} = \mathbf{L}(\vec{\beta}_{s1})\mathbf{e}_{f1} = E_{f0} \begin{bmatrix} \gamma_{s1} & -\gamma_{s1}\vec{\beta}_{s1}^T \\ -\gamma_{s1}\vec{\beta}_{s1} & \mathbf{I} + \kappa_{s1}\vec{\beta}_{s1}\vec{\beta}_{s1}^T \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = E_{f0} \begin{bmatrix} \gamma_{s1} \\ -\gamma_{s1}\vec{\beta}_{s1} \end{bmatrix}$$

From this, we may deduce that in the rest frame:

$$\gamma_{f0} = \gamma_{s1}, \quad \vec{\beta}_{f0} = -\vec{\beta}_{s1}.$$

Before we transform the force to the rest frame, we note that the quantities in the force four-vector all defined in terms of the lab frame. Before we proceed, let's see how these quantities relate to rest-frame observables.

$$\begin{aligned} \begin{bmatrix} r_{h1} \\ \vec{r}_{h1} \end{bmatrix} &= r_{h1} \begin{bmatrix} 1 \\ \hat{u}_1 \end{bmatrix} = \mathbf{L}(-\vec{\beta}_{s1})r_0 \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix} = r_0 \begin{bmatrix} \gamma_{s1}(1 + \vec{\beta}_{s1} \cdot \hat{u}_0) \\ \hat{u}_0 + \gamma_{s1}\vec{\beta}_{s1} + \kappa_{s1}\vec{\beta}_{s1}(\vec{\beta}_{s1} \cdot \hat{u}_0) \end{bmatrix} \\ \Rightarrow \hat{u}_1 &= \frac{\hat{u}_0 + \gamma_{s1}\vec{\beta}_{s1} + \kappa_{s1}\vec{\beta}_{s1}(\vec{\beta}_{s1} \cdot \hat{u}_0)}{\gamma_{s1}(1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} \\ \Rightarrow r_{h1} &= r_0 \gamma_{s1}(1 + \vec{\beta}_{s1} \cdot \hat{u}_0) = \gamma_{s1}(r_0 + \vec{\beta}_{s1} \cdot \vec{r}_0) \\ r_{h1}\vec{\beta}_{s1} &= \gamma_{s1}(1 + \vec{\beta}_{s1} \cdot \vec{r}_0)\vec{\beta}_{s1} \\ \Rightarrow \vec{r}_{h1} &= \vec{r}_0 + r_0\gamma_{s1}\vec{\beta}_{s1} + \kappa_{s1}\vec{\beta}_{s1}(\vec{\beta}_{s1} \cdot \vec{r}_0) \\ \Rightarrow \vec{r}_{r1} &= \vec{r}_{h1} - r_{h1}\vec{\beta}_{s1} = \vec{r}_0 + r_0\gamma_{s1}\vec{\beta}_{s1} + \kappa_{s1}\vec{\beta}_{s1}(\vec{\beta}_{s1} \cdot \vec{r}_0) - \gamma_{s1}(r_0 + \vec{\beta}_{s1} \cdot \vec{r}_0)\vec{\beta}_{s1} \\ &= \vec{r}_0 + r_0\gamma_{s1}\vec{\beta}_{s1} + \kappa_{s1}\vec{\beta}_{s1}(\vec{\beta}_{s1} \cdot \vec{r}_0) - \gamma_{s1}\vec{\beta}_{s1}r_0 - \gamma_{s1}(\vec{\beta}_{s1} \cdot \vec{r}_0)\vec{\beta}_{s1} \\ &= \vec{r}_0 + (\kappa_{s1} - \gamma_{s1})\vec{\beta}_{s1}(\vec{\beta}_{s1} \cdot \vec{r}_0) \\ &= \vec{r}_0 - \frac{\kappa_{s1}}{\gamma_{s1}}\vec{\beta}_{s1}(\vec{\beta}_{s1} \cdot \vec{r}_0) \quad \text{as } \kappa - \gamma = -\kappa/\gamma \end{aligned}$$

The projection of the ray line on the head line, defined as ρ in Fig. A.3 transforms as follows:

$$\begin{aligned}
\Rightarrow \rho_1 = \hat{u}_1 \cdot \vec{r}_{r1} &= \left[\frac{\hat{u}_0 + \gamma_{s1} \vec{\beta}_{s1} + \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \hat{u}_0)}{\gamma_{s1} (1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} \right] \cdot \left[\vec{r}_0 - \frac{\kappa_{s1}}{\gamma_{s1}} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0) \right] \\
&= \frac{1}{\gamma_{s1}^2 (1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} [\hat{u}_0 + \gamma_{s1} \vec{\beta}_{s1} + \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \hat{u}_0)] \cdot [\gamma_{s1} \vec{r}_0 - \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0)] \\
&= \frac{1}{\gamma_{s1}^2 (1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} [\gamma_{s1} (\hat{u}_0 \cdot \vec{r}_0) - \kappa_{s1} (\hat{u}_0 \cdot \vec{\beta}_{s1}) (\vec{\beta}_{s1} \cdot \vec{r}_0) + \gamma_{s1}^2 (\vec{\beta}_{s1} \cdot \vec{r}_0) - \gamma_{s1} \kappa_{s1} \vec{\beta}_{s1}^2 (\vec{\beta}_{s1} \cdot \vec{r}_0) \\
&\quad + \kappa_{s1} \gamma_{s1} (\vec{\beta}_{s1} \cdot \hat{u}_0) (\vec{\beta}_{s1} \cdot \vec{r}_0) - \kappa_{s1}^2 \beta_{s1}^2 (\vec{\beta}_{s1} \cdot \hat{u}_0) (\vec{\beta}_{s1} \cdot \vec{r}_0)] \\
&= \frac{1}{\gamma_{s1}^2 (1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} [\gamma_{s1} r_0 + \kappa_{s1} (-1 + \gamma_{s1} - \kappa_{s1} \beta_{s1}^2) (\hat{u}_0 \cdot \vec{\beta}_{s1}) (\vec{\beta}_{s1} \cdot \vec{r}_0) + \gamma_{s1} (-\kappa_{s1} \beta_{s1}^2 + \gamma_{s1}) (\vec{\beta}_{s1} \cdot \vec{r}_0)] \\
&\quad \text{collecting terms and using } \vec{r}_0 = r_0 \hat{u}_0 \\
&= \frac{1}{\gamma_{s1}^2 (1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} [\gamma_{s1} r_0 + \kappa_{s1} (-1 + \gamma_{s1} - \gamma_{s1} + 1) (\hat{u}_0 \cdot \vec{\beta}_{s1}) (\vec{\beta}_{s1} \cdot \vec{r}_0) + \gamma_{s1} (-\gamma_{s1} + 1 + \gamma_{s1}) (\vec{\beta}_{s1} \cdot \vec{r}_0)] \\
&\quad \text{as } \kappa \beta^2 = \gamma - 1 \\
&= \frac{1}{\gamma_{s1}^2 (1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} [\gamma_{s1} r_0 + \gamma_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0)] \\
&= \frac{1}{\gamma_{s1} (1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} [r_0 + (\vec{\beta}_{s1} \cdot r_0 \hat{u}_0)] \\
&= \frac{r_0 [1 + (\vec{\beta}_{s1} \cdot \hat{u}_0)]}{\gamma_{s1} (1 + \vec{\beta}_{s1} \cdot \hat{u}_0)} \\
&= \frac{r_0}{\gamma_{s1}}
\end{aligned}$$

This leads us to a useful conclusion:

$$(C.12) \quad r_0 = \gamma_s \rho.$$

It's very important to remember that the above expressions are for the **lab** quantities but in terms of rest-frame variables. The corresponding **rest-frame** quantities are:

$$\begin{aligned}
\hat{u}_1 &\rightarrow \hat{u}_0 \\
r_{h1} &\rightarrow r_{h0} = r_0 \\
\vec{r}_{h1} &\rightarrow \vec{r}_{h0} = \vec{r}_0 \\
\vec{r}_{r1} &\rightarrow \vec{r}_{r0} = \vec{r}_{h1} - \vec{0} = \vec{r}_0 \\
\rho_1 &\rightarrow \rho_0 = \hat{u}_0 \cdot \vec{r}_{r1} = r_0
\end{aligned}$$

Now we can write the lab-frame force in terms of these quantities:

$$\begin{aligned}
\vec{F}_1 &= \frac{q_f q_s}{4\pi\epsilon_0} \frac{\vec{r}_r}{\gamma_{s1}^2 \rho_1^3} = \frac{q_f q_s}{4\pi\epsilon_0} \frac{1}{\gamma_{s1}^2 (r_0 / \gamma_{s1})^3} \left[\vec{r}_0 - \frac{\kappa_{s1}}{\gamma_{s1}} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0) \right] \\
&= \frac{q_f q_s}{4\pi\epsilon_0} \frac{\gamma_{s1}}{r_0^3} \left[\vec{r}_0 - \frac{\kappa_{s1}}{\gamma_{s1}} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0) \right] \\
&= \frac{q_f q_s}{4\pi\epsilon_0} \frac{[\gamma_{s1} \vec{r}_0 - \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0)]}{r_0^3}
\end{aligned}$$

Now, let's transform this to the rest frame of the source:

$$\begin{aligned}
\mathbf{f}_0 &= \mathbf{L}(\vec{\beta}_{s1}) \mathbf{f}_1 = \begin{bmatrix} \gamma_{s1} & -\gamma_{s1} \vec{\beta}_{s1}^T \\ -\gamma_{s1} \vec{\beta}_{s1} & I + \kappa_{s1} \vec{\beta}_{s1} \vec{\beta}_{s1}^T \end{bmatrix} \begin{bmatrix} 0 \\ \vec{F}_1 \end{bmatrix} = \begin{bmatrix} -\gamma_{s1} (\vec{\beta}_{s1} \cdot \vec{F}_1) \\ \vec{F}_1 + \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{F}_1) \end{bmatrix} \\
\vec{F}_0 &= \frac{1}{\gamma_{f0}} [\vec{F}_1 + \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{F}_1)] \\
&= \frac{1}{\gamma_{s1}} [\vec{F}_1 + \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{F}_1)] \\
&= \frac{q_f q_s}{4\pi\epsilon_0 \gamma_{s1} r_0^3} [\gamma_{s1} \vec{r}_0 - \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0) + \kappa_{s1} \gamma_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0) - \kappa_{s1}^2 \vec{\beta}_{s1} \vec{\beta}_{s1}^2 (\vec{\beta}_{s1} \cdot \vec{r}_0)] \\
&= \frac{q_f q_s}{4\pi\epsilon_0 \gamma_{s1} r_0^3} [\gamma_{s1} \vec{r}_0 - \kappa_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0) + \kappa_{s1} \gamma_{s1} \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0) - \kappa_{s1} (\gamma_{s1} - 1) \vec{\beta}_{s1} (\vec{\beta}_{s1} \cdot \vec{r}_0)] \\
&= \frac{q_f q_s}{4\pi\epsilon_0 \gamma_{s1} r_0^3} [\gamma_{s1} \vec{r}_0]
\end{aligned}$$

So the force is finally:

$$(C.13) \quad \vec{F}_0 = \frac{q_f q_s}{4\pi\epsilon_0} \frac{\vec{r}_0}{r_0^3}$$

Note that this force is just the Coulomb force when both the source charge and field charge are at rest! That is, in the rest frame of the source, the motion of the field charge makes no difference at all.

C.5 Moving Field Charge, Moving Source Charge

With this result, we can do a Lorentz transformation $\mathbf{L}(-\vec{\beta}_{s2})$ to a frame \mathbf{R}_2 where the source moves with velocity $\vec{\beta}_{s2}$. In this frame both the source and field particles will be in motion, so it will give us the most general possible result.

First, let's construct the force four-vector in \mathbf{R}_0 .

$$\mathbf{f}_0 = \gamma_{f0} \begin{bmatrix} \vec{F}_0 \cdot \vec{\beta}_{f0} \\ \vec{F}_0 \end{bmatrix}$$

Then we will transform some of the important quantities into \mathbf{R}_2 .

Point S as the thread leaves the source

$$\mathbf{r}_{S2} = \begin{bmatrix} 0 \\ \vec{0} \end{bmatrix}$$

Point P as the thread arrives

$$\mathbf{r}_{P2} = \begin{bmatrix} ct_2 \\ \vec{r}_{h2} \end{bmatrix} = \begin{bmatrix} r_{h2} \\ \vec{r}_{h2} \end{bmatrix} = r_{h2} \begin{bmatrix} 1 \\ \hat{u}_2 \end{bmatrix}$$

Point U , the source when the thread arrives at P

$$\mathbf{r}_{U1} = r_{h2} \begin{bmatrix} 1 \\ \vec{\beta}_{s2} \end{bmatrix}$$

Source charge

$$\mathbf{e}_{s2} = \begin{bmatrix} E_{s2} \\ \vec{p}_{s2}c \end{bmatrix} = E_{s0}\gamma_{s2} \begin{bmatrix} 1 \\ \vec{\beta}_{s2} \end{bmatrix}$$

Field charge

$$\mathbf{e}_{f2} = \begin{bmatrix} E_{f2} \\ \vec{p}_{f2}c \end{bmatrix} = E_{f0}\gamma_{f2} \begin{bmatrix} 1 \\ \vec{\beta}_{f2} \end{bmatrix}$$

We have for the source charge:

$$\mathbf{e}_{s2} = E_{s0}\gamma_{s2} \begin{bmatrix} 1 \\ \vec{\beta}_{s2} \end{bmatrix} = \mathbf{L}(-\vec{\beta}_{s2})E_{s0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = E_{s0} \begin{bmatrix} \gamma_{s2} & +\gamma_{s2}\vec{\beta}_{s2}^T \\ +\gamma_{s2}\vec{\beta}_{s2} & \mathbf{I} + \kappa_{s2}\vec{\beta}_{s2}\vec{\beta}_{s2}^T \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = E_{s0}\gamma_{s2} \begin{bmatrix} 1 \\ \vec{\beta}_{s2} \end{bmatrix} \text{ as}$$

required.

For the field charge, we have:

$$\mathbf{e}_{f2} = E_{f0}\gamma_{f2} \begin{bmatrix} 1 \\ \vec{\beta}_{f2} \end{bmatrix} = E_{f0}\gamma_{f0} \begin{bmatrix} \gamma_{s2} & +\gamma_{s2}\vec{\beta}_{s2}^T \\ +\gamma_{s2}\vec{\beta}_{s2} & \mathbf{I} + \kappa_{s2}\vec{\beta}_{s2}\vec{\beta}_{s2}^T \end{bmatrix} \begin{bmatrix} 1 \\ \vec{\beta}_{f0} \end{bmatrix} = E_{f0}\gamma_{f0} \begin{bmatrix} \gamma_{s2}(1 + \vec{\beta}_{s2} \cdot \vec{\beta}_{f0}) \\ \vec{\beta}_{f0} + \gamma_{s2}\vec{\beta}_{s2} + \kappa_{s2}\vec{\beta}_{s2}(\vec{\beta}_{s2} \cdot \vec{\beta}_{f0}) \end{bmatrix}$$

and the other direction :

$$\mathbf{e}_{f0} = E_{f0}\gamma_{f0} \begin{bmatrix} 1 \\ \vec{\beta}_{f0} \end{bmatrix} = E_{f0}\gamma_{f2} \begin{bmatrix} \gamma_{s2} & -\gamma_{s2}\vec{\beta}_{s2}^T \\ -\gamma_{s2}\vec{\beta}_{s2} & \mathbf{I} + \kappa_{s2}\vec{\beta}_{s2}\vec{\beta}_{s2}^T \end{bmatrix} \begin{bmatrix} 1 \\ \vec{\beta}_{f2} \end{bmatrix} = E_{f0}\gamma_{f2} \begin{bmatrix} \gamma_{s2}(1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \\ \vec{\beta}_{f2} - \gamma_{s2}\vec{\beta}_{s2} + \kappa_{s2}\vec{\beta}_{s2}(\vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \end{bmatrix}$$

$$\Rightarrow \gamma_{f0} = \gamma_{f2}\gamma_{s2}(1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2})$$

$$\gamma_{f0}\vec{\beta}_{f0} = \gamma_{f2} \left[\vec{\beta}_{f2} - \gamma_{s2}\vec{\beta}_{s2} + \kappa_{s2}\vec{\beta}_{s2}(\vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \right] \quad \text{in terms of quantities in } \mathbf{R}_2.$$

Transforming \vec{r}_0 , we obtain:

$$\mathbf{r}_0 = r_0 \begin{bmatrix} 1 \\ \hat{\mathbf{u}}_0 \end{bmatrix} = r_{h2} \begin{bmatrix} \gamma_{s2} & -\gamma_{s2} \vec{\beta}_{s2}^T \\ -\gamma_{s2} \vec{\beta}_{s2} & \mathbf{I} + \kappa_{s2} \vec{\beta}_{s2} \vec{\beta}_{s2}^T \end{bmatrix} \begin{bmatrix} 1 \\ \hat{\mathbf{u}}_2 \end{bmatrix} = r_{h2} \begin{bmatrix} \gamma_{s2} (1 - \vec{\beta}_{s2} \cdot \hat{\mathbf{u}}_2) \\ \hat{\mathbf{u}}_2 - \gamma_{s2} \vec{\beta}_{s2} + \kappa_{s2} \vec{\beta}_{s2} (\vec{\beta}_{s2} \cdot \hat{\mathbf{u}}_2) \end{bmatrix}$$

$$\Rightarrow \vec{r}_0 = \vec{r}_{h2} - r_{h2} \gamma_{s2} \vec{\beta}_{s2} + \kappa_{s2} \vec{\beta}_{s2} (\vec{\beta}_{s2} \cdot \vec{r}_{h2}) \quad \text{in terms of quantities in } \mathbf{R}_2.$$

And the inverse of this transformation gives us:

$$\mathbf{r}_h = r_{h2} \begin{bmatrix} 1 \\ \hat{\mathbf{u}}_2 \end{bmatrix} = r_0 \begin{bmatrix} \gamma_{s2} & +\gamma_{s2} \vec{\beta}_{s2}^T \\ +\gamma_{s2} \vec{\beta}_{s2} & \mathbf{I} + \kappa_{s2} \vec{\beta}_{s2} \vec{\beta}_{s2}^T \end{bmatrix} \begin{bmatrix} 1 \\ \hat{\mathbf{u}}_0 \end{bmatrix} = r_0 \begin{bmatrix} \gamma_{s2} (1 + \vec{\beta}_{s2} \cdot \hat{\mathbf{u}}_0) \\ \hat{\mathbf{u}}_0 + \gamma_{s2} \vec{\beta}_{s2} + \kappa_{s2} \vec{\beta}_{s2} (\vec{\beta}_{s2} \cdot \hat{\mathbf{u}}_0) \end{bmatrix}$$

$$\Rightarrow \vec{r}_{h2} = \vec{r}_0 + r_0 \gamma_{s2} \vec{\beta}_{s2} + \kappa_{s2} \vec{\beta}_{s2} (\vec{\beta}_{s2} \cdot \vec{r}_0).$$

We'll also need the dot product $\vec{\beta}_{f0} \cdot \vec{r}_0$. This is most easily done in terms of Lorentz invariants. We know the four-vector dot-product of the head line and the field particle energy-momentum must be the same in both frames:

$$\begin{bmatrix} r_0 & \vec{r}_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} E_{of} \gamma_{f0} \\ E_{of} \gamma_{f0} \vec{\beta}_{f0} \end{bmatrix} = \begin{bmatrix} r_h & \vec{r}_h \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} E_{of} \gamma_{f2} \\ E_{of} \gamma_{f2} \vec{\beta}_{f2} \end{bmatrix}$$

$$r_0 E_{of} \gamma_{f0} - E_{of} \gamma_{f0} (\vec{r}_0 \cdot \vec{\beta}_{f0}) = r_h E_{of} \gamma_{f2} - E_{of} \gamma_{f2} (\vec{r}_h \cdot \vec{\beta}_{f2})$$

$$r_0 \gamma_{f0} - \gamma_{f0} (\vec{r}_0 \cdot \vec{\beta}_{f0}) = r_h \gamma_{f2} - \gamma_{f2} (\vec{r}_h \cdot \vec{\beta}_{f2})$$

$$\gamma_{f0} (\vec{r}_0 \cdot \vec{\beta}_{f0}) = r_0 \gamma_{f0} - r_h \gamma_{f2} + \gamma_{f2} (\vec{r}_h \cdot \vec{\beta}_{f2})$$

$$\vec{r}_0 \cdot \vec{\beta}_{f0} = r_0 - \frac{\gamma_{f2}}{\gamma_{f0}} (r_h - \vec{r}_h \cdot \vec{\beta}_{f2})$$

So now, let's transform the force four-vector:

$$\mathbf{f}_2 = \gamma_{f2} \begin{bmatrix} \vec{F}_2 \cdot \vec{\beta}_{f2} \\ \vec{F}_2 \end{bmatrix} = \mathbf{L}(-\vec{\beta}_{s2}) \gamma_{f0} \begin{bmatrix} \vec{F}_0 \cdot \vec{\beta}_{f0} \\ \vec{F}_0 \end{bmatrix} = \gamma_{f0} \begin{bmatrix} \gamma_{s2} & +\gamma_{s2} \vec{\beta}_{s2}^T \\ +\gamma_{s2} \vec{\beta}_{s2} & \mathbf{I} + \kappa_{s2} \vec{\beta}_{s2} \vec{\beta}_{s2}^T \end{bmatrix} \begin{bmatrix} \vec{F}_0 \cdot \vec{\beta}_{f0} \\ \vec{F}_0 \end{bmatrix}$$

$$= \gamma_{f0} \begin{bmatrix} \gamma_{s2} & +\gamma_{s2} \vec{\beta}_{s2}^T \\ +\gamma_{s2} \vec{\beta}_{s2} & \mathbf{I} + \kappa_{s2} \vec{\beta}_{s2} \vec{\beta}_{s2}^T \end{bmatrix} \begin{bmatrix} \vec{F}_0 \cdot \vec{\beta}_{f0} \\ \vec{F}_0 \end{bmatrix} = \gamma_{f0} \begin{bmatrix} \gamma_{s2} (\vec{F}_0 \cdot \vec{\beta}_{f0} - \vec{\beta}_{s2} \cdot \vec{F}_0) \\ \vec{F}_0 + \gamma_{s2} (\vec{F}_0 \cdot \vec{\beta}_{f0}) \vec{\beta}_{s2} + \kappa_{s2} \vec{\beta}_{s2} (\vec{\beta}_{s2} \cdot \vec{F}_0) \end{bmatrix}$$

$$\vec{F}_2 = \frac{\gamma_{f0}}{\gamma_{f2}} \left[\vec{F}_0 + \gamma_{s2} (\vec{F}_0 \cdot \vec{\beta}_{f0}) \vec{\beta}_{s2} + \kappa_{s2} \vec{\beta}_{s2} (\vec{\beta}_{s2} \cdot \vec{F}_0) \right]$$

$$= \frac{q_s q_f}{4\pi\epsilon_0 r_0^3} \frac{\gamma_{f0}}{\gamma_{f2}} \left[\vec{r}_0 + \gamma_{s2} (\vec{r}_0 \cdot \vec{\beta}_{f0}) \vec{\beta}_{s2} + \kappa_{s2} (\vec{r}_0 \cdot \vec{\beta}_{s2}) \vec{\beta}_{s2} \right]$$

$$= \frac{q_s q_f}{4\pi\epsilon_0 r_0^3} \frac{\gamma_{f0}}{\gamma_{f2}} \left[\vec{r}_0 + \kappa_{s2} (\vec{r}_0 \cdot \vec{\beta}_{s2}) \vec{\beta}_{s2} + \gamma_{s2} (\vec{r}_0 \cdot \vec{\beta}_{f0}) \vec{\beta}_{s2} \right]$$

From the equation for \vec{r}_{h2} above, we note that we can combine the first and second terms in the square brackets:

$$\begin{aligned}\vec{r}_{h2} - r_0 \gamma_{s2} \vec{\beta}_{s2} &= \vec{r}_0 + \kappa_{s2} \vec{\beta}_{s2} (\vec{\beta}_{s2} \cdot \vec{r}_0) \\ \Rightarrow \vec{F}_2 &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{r_0^3} \frac{\gamma_{f0}}{\gamma_{f2}} \left[\vec{r}_{h2} + r_0 \gamma_{s2} \vec{\beta}_{s2} + \gamma_{s2} (\vec{r}_0 \cdot \vec{\beta}_{f0}) \vec{\beta}_{s2} \right]\end{aligned}$$

From Eq. (C.12):

$$r_0 = \gamma_{s2} \rho_2$$

Using this and the expression for γ_{f2} above, we can further simplify the force:

$$\vec{F}_2 = \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^3 \rho_2^3} \gamma_{s2} (1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \left[\vec{r}_{h2} - r_0 \gamma_{s2} \vec{\beta}_{s2} + \gamma_{s2} (\vec{r}_0 \cdot \vec{\beta}_{f0}) \vec{\beta}_{s2} \right]$$

Now we insert the equation for $\vec{\beta}_{f0} \cdot \vec{r}_0$:

$$\begin{aligned}\vec{F}_2 &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^3 \rho_2^3} \gamma_{s2} (1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \left\{ \vec{r}_{h2} - r_0 \gamma_{s2} \vec{\beta}_{s2} + \gamma_{s2} \left[r_0 - \frac{\gamma_{f2}}{\gamma_{f0}} (r_{h2} - \vec{r}_{h2} \cdot \vec{\beta}_{f2}) \right] \vec{\beta}_{s2} \right\} \\ &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^3 \rho_2^3} \gamma_{s2} (1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \left\{ \vec{r}_{h2} - r_0 \gamma_{s2} \vec{\beta}_{s2} + \gamma_{s2} r_0 \vec{\beta}_{s2} - \gamma_{s2} \frac{\gamma_{f2}}{\gamma_{f0}} (r_{h2} - \vec{r}_{h2} \cdot \vec{\beta}_{f2}) \vec{\beta}_{s2} \right\} \\ &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^3 \rho_2^3} \gamma_{s2} (1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \left\{ \vec{r}_{h2} - \gamma_{s2} \frac{\gamma_{f2}}{\gamma_{f0}} (r_{h2} - \vec{r}_{h2} \cdot \vec{\beta}_{f2}) \vec{\beta}_{s2} \right\} \\ &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^3 \rho_2^3} \gamma_{s2} (1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \left\{ \vec{r}_{h2} - \gamma_{s2} \frac{1}{\gamma_{s2} (1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2})} (r_{h2} - \vec{r}_{h2} \cdot \vec{\beta}_{f2}) \vec{\beta}_{s2} \right\}\end{aligned}$$

$$\begin{aligned}\vec{F}_2 &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^3 \rho_2^3} \gamma_{s2} (1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) \left\{ \vec{r}_{h2} - \frac{1}{(1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2})} (r_{h2} - \vec{r}_{h2} \cdot \vec{\beta}_{f2}) \vec{\beta}_{s2} \right\} \\ &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^2 \rho_2^3} \left\{ \vec{r}_{h2} (1 - \vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) - (r_{h2} - \vec{r}_{h2} \cdot \vec{\beta}_{f2}) \vec{\beta}_{s2} \right\} \\ &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^2 \rho_2^3} \left\{ \vec{r}_{h2} - \vec{r}_{h2} (\vec{\beta}_{s2} \cdot \vec{\beta}_{f2}) - r_{h2} \vec{\beta}_{s2} + (\vec{r}_{h2} \cdot \vec{\beta}_{f2}) \vec{\beta}_{s2} \right\} \\ &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^2 \rho_2^3} \left\{ \vec{r}_{h2} - r_{h2} \vec{\beta}_{s2} + \vec{\beta}_{f2} \times (\vec{\beta}_{s2} \times \vec{r}_{h2}) \right\} \quad \text{BAC minus CAB rule} \\ &= \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_{s2}^2 \rho_2^3} \left\{ \vec{r}_{r2} + \vec{\beta}_{f2} \times (\vec{\beta}_{s2} \times \vec{r}_{h2}) \right\} \quad \text{as } \vec{r}_{r2} = \vec{r}_{h2} - r_{h2} \vec{\beta}_{s2}\end{aligned}$$

One last thing to note is that

$$\vec{\beta}_{f2} \times (\vec{\beta}_{s2} \times \vec{r}_{r2}) = \vec{\beta}_{f2} \times [\vec{\beta}_{s2} \times (\vec{r}_{h2} - r_{h2} \vec{\beta}_{s2})] = \vec{\beta}_{f2} \times (\vec{\beta}_{s2} \times \vec{r}_{h2}) \quad \text{since } \vec{\beta}_{s2} \times \vec{\beta}_{s2} = 0.$$

Using this result and dropping all the “2” subscripts which are no longer needed as everything is in the \mathbf{R}_2 frame, we can write our final result as:

$$(C.14) \quad \vec{F} = \frac{q_s q_f}{4\pi\epsilon_0} \frac{1}{\gamma_s^2 \rho^3} \left[\vec{r}_r + \vec{\beta}_f \times (\vec{\beta}_s \times \vec{r}_r) \right]$$

As before, we can define the electric and magnetic fields as:

$$(C.15) \quad \begin{aligned} \vec{E} &= \frac{q_s}{4\pi\epsilon_0} \frac{\vec{r}_r}{\gamma_s^2 \rho^3} \\ \vec{B} &= \frac{1}{c} \vec{\beta}_s \times \vec{E}. \end{aligned}$$

In terms of the fields, the force is:

$$(C.16) \quad \vec{F} = q_f \vec{E} + q_f c \vec{\beta}_f \times \vec{B} = q_f \vec{E} + q_f \vec{v}_f \times \vec{B},$$

which is the Lorentz force law.

We can also define the fields in terms of threads and stubs as:

$$\begin{aligned} \vec{E} &= \frac{e}{\epsilon_0} v \vec{\ell} \\ \vec{B} &= \frac{1}{c} \frac{e}{\epsilon_0} v \vec{s} \end{aligned}$$

so that the magnetic field is:

$$\begin{aligned} \vec{B} &= \frac{1}{c} \vec{\beta}_s \times \frac{e}{\epsilon_0} v \vec{\ell} \\ &= \frac{1}{c} \frac{e}{\epsilon_0} v (\vec{\beta}_s \times \vec{\ell}) \end{aligned}$$

This suggests that the stub equals:

$$(C.17) \quad \vec{s} = \vec{\beta}_s \times \vec{\ell}$$

In the text, we wrote the stub as $\vec{s} = \hat{r}_h \times \vec{\ell}$. We see that this is equivalent *for sources moving at constant velocity* because:

$$\begin{aligned}
\vec{s} &= \hat{r}_h \times \vec{\ell} \\
&= \frac{\vec{r}_h}{r_h} \times \vec{\ell} \\
&= \frac{1}{r_h} (\vec{r}_r + r_h \vec{\beta}_s) \times \vec{\ell} \\
&= \vec{\beta}_s \times \vec{\ell} \quad \text{as } \vec{\ell} \text{ and } \vec{r}_r \text{ are parallel.}
\end{aligned}$$

C.6 Velocity and Acceleration Threads

Now we want to ask what happens when the source charge is accelerating. Let's begin in the rest frame of the source at time $ct_0 = 0$. Since the source is accelerating, it will only be at rest in this frame instantaneously, however. The time it takes for the thread to be emitted in this frame is $\Delta ct_0 = \ell_0$. As long as we let the thread length $\ell_0 \rightarrow 0$, we don't have to worry about how the source charge's clocks are slowing down during the emission process.

If we look at this same process in the lab frame, the time interval becomes $\Delta ct = \gamma \ell_0$, as in the last section. In this time, the velocity changes by $\Delta \vec{\beta} = \vec{\alpha} \Delta ct = \vec{\alpha} \gamma \ell_0$ and the source moves a distance $\Delta \vec{r} = \vec{\beta} \Delta ct + \frac{1}{2} \vec{\alpha} (\Delta ct)^2 = \vec{\beta} \gamma \ell_0 + \frac{1}{2} \vec{\alpha} (\gamma \ell_0)^2 \approx \vec{\beta} \gamma \ell_0$. Since ℓ_0 goes to zero in the limit, ℓ_0^2 goes to zero much more rapidly. Thus we can safely ignore the last term as it involves the square of an infinitesimal quantity. In this section we will frequently ignore terms with a product of two or more infinitesimal quantities. Notice that this means that the tail is emitted from essentially the same point as if there were no acceleration.

In fact, threads emitted from an accelerating source charge are similar in almost all ways to those emitted from a source moving at constant velocity. They are emitted isotropically in the instantaneous rest frame of the source, just as when the source travels at constant velocity. The density of threads emitted is the same as for constant velocity. The one thing that is different – and the difference is very important – is that the tail goes off at a different angle than the head. While this seems rather insignificant, it means that the thread grows in length and changes direction as it travels away from the source. This is discussed in more detail in Lesson 10 of the text. What we wish to find in this section is how acceleration changes the direction of the tail with respect to the head. Although the question and answer are both quite simple, the intermediate steps are rather complicated.

Before we continue, we find that it's useful to define a “velocity thread” and an “acceleration thread.” In Fig. C.3 we have modified Fig C.1 for a source accelerating in the $+x$ direction. The tail line for the constant velocity case is labeled “original tail line.” As discussed above, the tail line for the accelerating source comes off from point T , but at a different angle than the head. It is labeled “tail line.” The thread, as always is the vector that joins the tail to the

head. However, we can think of the thread as the sum of two contributions. The first contribution is the thread that we would have had if there were no acceleration. This is called the “velocity head,” as shown in the figure. The vector that joins the tail of the thread to the tail of the velocity thread is called the “acceleration thread.” This represents the additional contribution to the thread (end hence to the electric field) produced by the source charge’s acceleration. The constant velocity formulas we previously derived are valid for the velocity thread. Now we only need to add to that some new formulas for the acceleration thread.

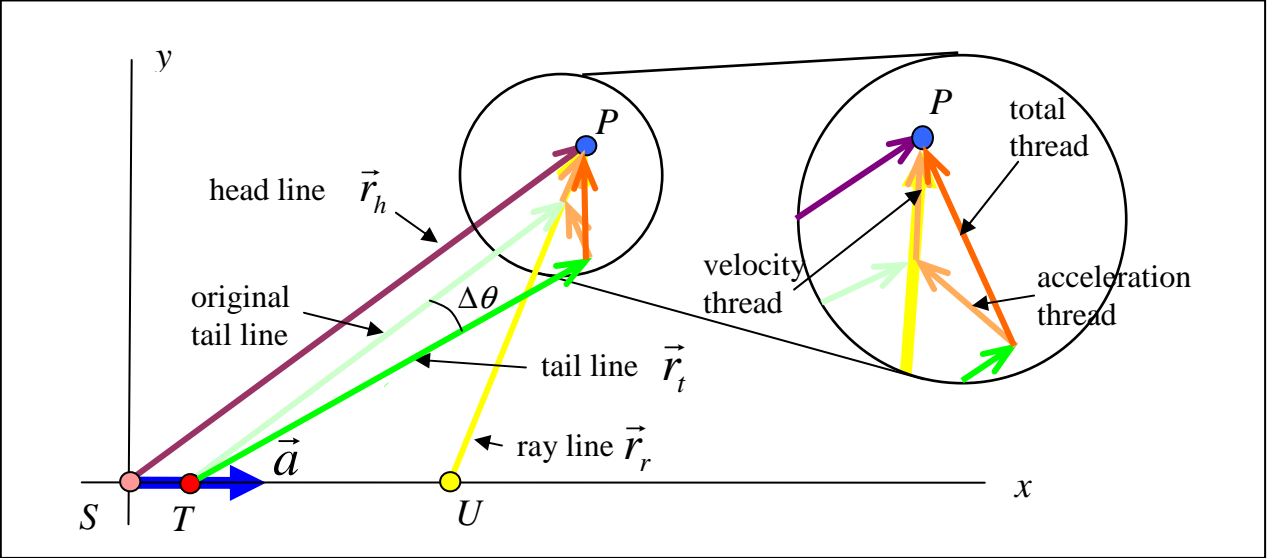


Figure C.3. The velocity and acceleration threads.

In other words, we can save ourselves a significant amount of work by breaking the thread into two components: the constant velocity component and the acceleration component.

C.7 Acceleration Threads and Fields

In the text we discussed some of the qualitative aspects of what acceleration does to a thread. As depicted in Fig. C.4, we can picture, in a very rough, qualitative sense, the acceleration vector as a force applied to the tail of the thread. We can view it as pulling the tail in one direction or the other. Note in particular that if a thread is parallel to the direction of the acceleration, the tail must come off in the same direction as the head.

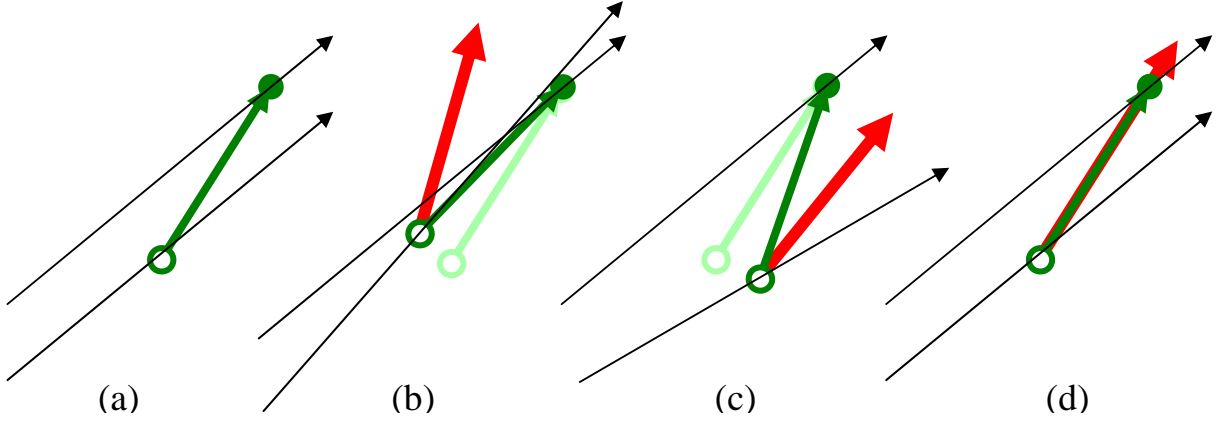


Figure C.4. How acceleration affects the direction of threads. The long, black arrows indicate the head and tail directions. The thick red arrow is the acceleration direction. (a) There is no acceleration, so the head and tail directions are the same. (b) The acceleration pulls the tail to the left. (The thread with no acceleration is shown in light green for comparison.) (c) The acceleration pulls the tail to the right. (d) The acceleration leaves the tail direction unchanged.

If the head of a given thread is emitted in a direction \hat{u}_h , the tail will be generally be emitted in another direction, \hat{u}_t . Since Δct is small, the difference in these two directions must also be small. We may therefore write $\hat{u}_t = \hat{u}_h + \Delta\vec{u}$. We know that the magnitude of each unit vector is one, so

$$\begin{aligned} \hat{u}_t \cdot \hat{u}_t &= (\hat{u}_h + \Delta\vec{u})^2 = \hat{u}_h \cdot \hat{u}_h + 2\hat{u}_h \cdot \Delta\vec{u} + \Delta\vec{u} \cdot \Delta\vec{u} \\ 1 &= 1 + 2\hat{u}_h \cdot \Delta\vec{u} \quad \text{dropping the last term as it very small. is small} \\ \Rightarrow \hat{u}_h \cdot \Delta\vec{u} &= 0 \end{aligned}$$

All this says is that when we add a small $\Delta\vec{u}$ to \hat{u}_h , the length of the vector remains the same so $\Delta\vec{u}$ must be perpendicular to \hat{u}_h . Since adding $\Delta\vec{u}$ changes the vector's direction without changing its length, we can describe this transformation as a rotation.

In particular, the process of accelerating a source initially at rest can cause the tail direction of every head to be different than the head direction. However, the rotation that relates the head and tail directions of one thread will be the same as the rotation that relates the head and tail directions of all other threads emitted at this same time. So in the frame where the source is at rest when the head is emitted, the acceleration effectively adds a term $\Delta\vec{u}_0$ to \hat{u}_0 or equivalently rotates \hat{u}_0 by a small amount. We write this as:

$$\begin{bmatrix} 1 \\ \hat{u}'_0 \end{bmatrix} = \mathbf{R}_r \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix}$$

where

\hat{u}_0 is a unit vector in the direction of the head in the head rest frame.

\hat{u}'_0 is a unit vector in the direction of the tail in the head rest frame.

General rotations in three dimensions are really quite complicated. Our rotations, however, involve only infinitesimal angles, so they are much simpler. Let's begin with a general rotation by an angle θ about the z axis. The general form is:

$$\mathbf{R}_{3z} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For small angles $\Delta\theta$, we know that $\sin \Delta\theta \approx \Delta\theta$ and $\cos \Delta\theta \approx 1$ ignoring terms of order $(\Delta\theta)^2$. The rotation then becomes:

$$\mathbf{R}_{3z} = \begin{bmatrix} 1 & -\Delta\theta & 0 \\ \Delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A general rotation can be thought of as a rotation about each of the x , y , and z axes in sequence. For infinitesimal rotations, unlike rotations of large angles, the order of the rotations does not matter. The rotation matrix becomes:

$$\mathbf{R}_3 = \begin{bmatrix} 1 & -s_z & s_y \\ s_z & 1 & s_x \\ -s_y & s_x & 1 \end{bmatrix}$$

where s_x , s_y , and s_z are infinitesimal rotation angles about the x , y , and z axes, respectively.

Now let's apply this rotation matrix to an arbitrary vector:

$$\mathbf{R}_3 \vec{a} = \begin{bmatrix} 1 & -s_z & s_y \\ s_z & 1 & s_x \\ -s_y & s_x & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} a_x + s_y a_z - s_z a_y \\ a_y + s_z a_x - s_x a_z \\ a_z + s_x a_y - s_y a_x \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \begin{bmatrix} s_y a_z - s_z a_y \\ s_z a_x - s_x a_z \\ s_x a_y - s_y a_x \end{bmatrix} = \vec{a} + \vec{s} \times \vec{a}$$

where $\vec{s} \equiv \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$.

Note how infinitesimal rotations are simply related to cross products. Also note the similarity of this to the treatment of the field strength tensor in Sect. B.7.

For convenience, we can consider the infinitesimal rotation matrix to be the sum of two other matrices:

$$\mathbf{R}_3 = \begin{bmatrix} 1 & -s_z & s_y \\ s_z & 1 & -s_x \\ -s_y & s_x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix} \equiv \mathbf{I} + \mathbf{S}_3,$$

$$\mathbf{S}_3 \vec{a} = \vec{s} \times \vec{a}.$$

If we wish to rotate four-vectors, we have a very similar rotation matrix, since rotations do not affect the time-components of four-vectors:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -s_z & s_y \\ 0 & s_z & 1 & -s_x \\ 0 & -s_y & s_x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -s_z & s_y \\ 0 & s_z & 0 & -s_x \\ 0 & -s_y & s_x & 0 \end{bmatrix} = \mathbf{I}_4 + \mathbf{S}$$

Returning to the rest-frame rotation, we have:

$$\begin{bmatrix} 1 \\ \hat{u}'_0 \end{bmatrix} = \mathbf{R}_r \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix} = (\mathbf{I}_4 + \mathbf{S}) \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix} + \mathbf{S} \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix}$$

$$\Rightarrow \vec{s} \times \hat{u}_0 = \Delta \vec{u}_0.$$

In the end, we wish to solve for \vec{s} , but first let's do a little more math with the Lorentz transformations and the rotation. Let's consider a general case where the thread's direction four-vector is in an arbitrary direction. We know if we start with the direction vector for the head in the lab frame, transform to the rest frame of the head, divide by a normalization factor N , rotate the direction four-vector in the rest frame to account for the transformation from the rest frame of the head to the rest frame of the tail, apply a Lorentz transformation back to the lab, normalize the resultant vector once again, and finally rotate the unit vector in the lab frame, we end up with the direction four-vector with which we started. Mathematically, we may write:

$$(C.18) \quad \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = \mathbf{R}_l \frac{1}{N'} \mathbf{L}(-\vec{\beta}') \mathbf{R}_r \frac{1}{N} \mathbf{L}(\vec{\beta}) \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix}$$

Just to belabor the point, let's go through this process step by step:

$$\begin{aligned}\begin{bmatrix} u_0 \\ \bar{u}_0 \end{bmatrix} &= \mathbf{L}(\vec{\beta}) \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} \\ \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix} &= \frac{1}{u_0} \begin{bmatrix} u_0 \\ \bar{u}_0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ \hat{u}'_0 \end{bmatrix} &= \mathbf{R}_r \begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} u' \\ \bar{u}' \end{bmatrix} &= \mathbf{L}(-\vec{\beta}') \begin{bmatrix} 1 \\ \hat{u}'_0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ \hat{u}' \end{bmatrix} &= \frac{1}{u'} \begin{bmatrix} u' \\ \bar{u}' \end{bmatrix} \\ \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} &= \mathbf{R}_l \begin{bmatrix} 1 \\ \hat{u}' \end{bmatrix}\end{aligned}$$

We know the Lorentz transformations, but we don't know either of the rotations. What we really want to know is the lab rotation matrix \mathbf{R}_l . To find this, however, we must first know the rest frame rotation matrix \mathbf{R}_r . Note that \mathbf{R}_r just accounts for the transformation from the head's rest frame to the tail's rest frame, and must depend only on $\vec{\beta}$ and $\vec{\beta}'$, not on \hat{u} . As we mentioned above, for given values of $\vec{\beta}$ and $\vec{\beta}'$, if we can find \mathbf{R}_r for any value of \hat{u} , we have found it for all values of \hat{u} .

From Fig. C.1, we know that the head and ray lines are related by the vector equation:

$$\vec{r}_h = r_h \vec{\beta} + \vec{r}_r$$

Dividing by r_h , we obtain:

$$\hat{u} = \vec{\beta} + \frac{\vec{r}_r}{r_h}.$$

Since the thread is along the direction of the ray line, we take $\Delta\vec{\beta}$ to be in this direction as well.

$$\begin{aligned}\text{Let } \frac{\vec{r}_r}{r_h} &\equiv \xi \Delta\vec{\beta} \\ \text{so } \xi &= \frac{r_r}{r_h \Delta\beta} \\ \Rightarrow \hat{u} &= \vec{\beta} + \xi \Delta\vec{\beta}\end{aligned}$$

To find the direction of the head in the rest frame of the head, we apply a Lorentz transformation to the thread direction (see Sect. B.6.)

$$\mathbf{L}(\vec{\beta}) \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = \begin{bmatrix} \gamma(1 - \vec{\beta} \cdot \hat{u}) \\ \hat{u} - \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u}) \end{bmatrix} = \begin{bmatrix} u_0 \\ \vec{u}_0 \end{bmatrix}$$

$$\hat{u}_0 = \frac{\vec{u}_0}{u_0} = \frac{\hat{u} - \gamma\vec{\beta} + \kappa\vec{\beta}(\vec{\beta} \cdot \hat{u})}{\gamma(1 - \vec{\beta} \cdot \hat{u})}$$

We can rearrange this so it's in a more convenient form:

$$\begin{bmatrix} 1 \\ \hat{u}_0 \end{bmatrix} = \frac{1}{N} \mathbf{L}(\vec{\beta}) \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} \quad \text{where } N = \gamma(1 - \vec{\beta} \cdot \hat{u})$$

As it turns out, we won't really need to evaluate ξ , but it is important to know that ξ is a fixed number that we could find if we wanted. It is also important to note that although $\Delta\vec{\beta}$ itself is infinitesimal, $\xi\Delta\vec{\beta}$ is not.

Now we need to get back to the rotation matrices. If we let $\hat{u} = \vec{\beta} + \xi\Delta\vec{\beta}$, we know that the head and tail directions are the same, so $\mathbf{R}_l = \mathbf{I}_4$. Then Eq. (C.18) becomes:

$$(C.19) \quad \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = \frac{1}{N'} \mathbf{L}(-\vec{\beta}') \mathbf{R}_r \frac{1}{N} \mathbf{L}(\vec{\beta}) \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix}, \quad \hat{u} = \vec{\beta} + \xi\Delta\vec{\beta}$$

We could put the elements in the matrices and multiply everything out, but the algebra involved is horrific. (Yes, I tried it... several ways!) But if we're clever, we can save ourselves this work.

First, let's put all the operators on the left and all the numbers on the right:

$$\mathbf{L}(-\vec{\beta}') \mathbf{R}_r \mathbf{L}(\vec{\beta}) \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = \begin{bmatrix} U \\ \vec{U} \end{bmatrix} = NN' \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix}, \quad \frac{1}{U} \begin{bmatrix} U \\ \vec{U} \end{bmatrix} = \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix}.$$

Remember that it's a special characteristic of direction vectors for threads (or photons or anything traveling at the speed of light) that the spatial component is a unit vector and the time component is 1. Normally we can't divide a four-vector by the time component u_0 and expect the spatial component to be a unit vector, as we did above. But since we can, we don't really need to know N and N' in advance. We can just multiply the matrices out and know that $NN' = U$.

Let's define $\mathbf{M} \equiv \mathbf{L}(-\vec{\beta}') \mathbf{R}_r \mathbf{L}(\vec{\beta})$. Then the spatial part of $\mathbf{M} \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix}$ must be parallel to \hat{u} .

To evaluate this expression, let's first write everything in terms of infinitesimals:

$$\mathbf{M} = [\mathbf{L}(-\vec{\beta}) + \Delta\mathbf{L}(-\vec{\beta})](\mathbf{I}_4 + \mathbf{S}_r)\mathbf{L}(\vec{\beta})$$

Now, let's simplify this expression by keeping only the terms that have no more than one factor of the infinitesimal matrices \mathbf{S} and $\Delta\mathbf{L}$.

$$\begin{aligned} \mathbf{M} &= \mathbf{L}(-\vec{\beta})\mathbf{I}_4\mathbf{L}(\vec{\beta}) + \mathbf{L}(-\vec{\beta})\mathbf{S}_r\mathbf{L}(\vec{\beta}) + \Delta\mathbf{L}(-\vec{\beta})\mathbf{I}_4\mathbf{L}(\vec{\beta}) \\ &= \mathbf{I}_4 + \mathbf{L}(-\vec{\beta})\mathbf{S}_r\mathbf{L}(\vec{\beta}) + \Delta\mathbf{L}(-\vec{\beta})\mathbf{L}(\vec{\beta}) \end{aligned}$$

Let's evaluate the matrix products explicitly:

$$\begin{aligned} \mathbf{S}_r\mathbf{L}(\vec{\beta}) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -s_z & s_y \\ 0 & s_z & 0 & -s_x \\ 0 & -s_y & s_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + \kappa\beta_x^2 & \kappa\beta_x\beta_y & \kappa\beta_x\beta_z \\ -\gamma\beta_y & \kappa\beta_x\beta_y & 1 + \kappa\beta_y^2 & \kappa\beta_y\beta_z \\ -\gamma\beta_z & \kappa\beta_x\beta_z & \kappa\beta_y\beta_z & 1 + \kappa\beta_z^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\gamma(\beta_z s_y - \beta_y s_z) & \kappa\beta_x(\beta_z s_y - \beta_y s_z) & -s_z + \kappa\beta_y(\beta_z s_y - \beta_y s_z) & s_y + \kappa\beta_z(\beta_z s_y - \beta_y s_z) \\ -\gamma(\beta_x s_z - \beta_z s_x) & s_z + \kappa\beta_x(\beta_x s_z - \beta_z s_x) & \kappa\beta_y(\beta_x s_z - \beta_z s_x) & -s_x + \kappa\beta_z(\beta_x s_z - \beta_z s_x) \\ -\gamma(\beta_y s_x - \beta_x s_y) & -s_y + \kappa\beta_x(\beta_y s_x - \beta_x s_y) & s_x + \kappa\beta_y(\beta_y s_x - \beta_x s_y) & \kappa\beta_z(\beta_y s_x - \beta_x s_y) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\gamma C_x & \kappa\beta_x C_x & \kappa\beta_y C_x & \kappa\beta_z C_x \\ -\gamma C_y & \kappa\beta_x C_y & \kappa\beta_y C_y & \kappa\beta_z C_y \\ -\gamma C_z & \kappa\beta_x C_z & \kappa\beta_y C_z & \kappa\beta_z C_z \end{bmatrix} + \mathbf{S}_r \quad \text{where } \vec{C} \equiv \vec{s} \times \vec{\beta} \\ &= \begin{bmatrix} 0 & \vec{0}^T \\ -\gamma\vec{C} & \kappa\vec{C}\vec{\beta}^T \end{bmatrix} + \mathbf{S}_r \end{aligned}$$

$$\begin{aligned} \mathbf{L}(-\vec{\beta})\mathbf{S}_r &= [\mathbf{S}_r^T\mathbf{L}(-\vec{\beta})]^T = [-\mathbf{S}_r\mathbf{L}(-\vec{\beta})]^T = -\left(\begin{bmatrix} 0 & \vec{0}^T \\ +\gamma\vec{C} & +\kappa\vec{C}\vec{\beta}^T \end{bmatrix} + \mathbf{S}_r \right)^T = \begin{bmatrix} 0 & -\gamma\vec{C}^T \\ \vec{0} & -\kappa\vec{\beta}\vec{C}^T \end{bmatrix} - \mathbf{S}_r^T \\ &= \begin{bmatrix} 0 & -\gamma\vec{C}^T \\ \vec{0} & -\kappa\vec{\beta}\vec{C}^T \end{bmatrix} + \mathbf{S}_r \end{aligned}$$

$$\text{as } (\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T, \quad \mathbf{L}^T = \mathbf{L}, \quad \mathbf{S}^T = -\mathbf{S}, \quad (\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$\begin{aligned}
\mathbf{L}(-\vec{\beta})\mathbf{S}_r^T\mathbf{L}(\vec{\beta}) &= \begin{bmatrix} \gamma & +\gamma\vec{\beta}^T \\ +\gamma\vec{\beta} & \mathbf{I} + \kappa\vec{\beta}\vec{\beta}^T \end{bmatrix} \begin{bmatrix} 0 & \vec{0}^T \\ -\gamma\vec{C} & \kappa\vec{C}\vec{\beta}^T \end{bmatrix} + \mathbf{L}(-\vec{\beta})\mathbf{S}_r \\
&= \begin{bmatrix} -\gamma^2(\vec{\beta}\cdot\vec{C}) & \kappa\gamma(\vec{\beta}\cdot\vec{C})\vec{\beta}^T \\ -\gamma\vec{C} - \kappa\gamma(\vec{\beta}\cdot\vec{C})\vec{\beta} & \kappa\vec{C}\vec{\beta}^T + \kappa^2(\vec{\beta}\cdot\vec{C})\vec{\beta}\vec{\beta}^T \end{bmatrix} + \begin{bmatrix} 0 & -\gamma\vec{C}^T \\ \vec{0} & -\kappa\vec{\beta}\vec{C}^T \end{bmatrix} + \mathbf{S}_r \\
&= \begin{bmatrix} -\gamma^2(\vec{\beta}\cdot\vec{C}) & -\gamma\vec{C}^T + \kappa\gamma(\vec{\beta}\cdot\vec{C})\vec{\beta}^T \\ -\gamma\vec{C} - \kappa\gamma(\vec{\beta}\cdot\vec{C})\vec{\beta} & \kappa\vec{C}\vec{\beta}^T - \kappa\vec{\beta}\vec{C}^T + \kappa^2(\vec{\beta}\cdot\vec{C})\vec{\beta}\vec{\beta}^T \end{bmatrix} + \mathbf{S}_r
\end{aligned}$$

This can be simplified considerably by noting that $\vec{\beta}\cdot\vec{C} = \vec{\beta}\cdot(\vec{s}\times\vec{\beta}) = 0$ by the dot-cross rule:

$$(C.20) \quad \mathbf{L}(-\vec{\beta})\mathbf{S}_r^T\mathbf{L}(\vec{\beta}) = \begin{bmatrix} 0 & -\gamma\vec{C}^T \\ -\gamma\vec{C} & \kappa\vec{C}\vec{\beta}^T - \kappa\vec{\beta}\vec{C}^T \end{bmatrix} + \mathbf{S}_r$$

Next, we need to find $\Delta\mathbf{L}(-\vec{\beta})$. This is a bit complicated because \mathbf{L} is a matrix, but the process is really just differentiation. Using the product rule on each component of \mathbf{L} , we have:

$$\begin{aligned}
\mathbf{L}(-\vec{\beta}) &= \begin{bmatrix} \gamma & \gamma\vec{\beta}^T \\ \gamma\vec{\beta} & \mathbf{I} + \kappa\vec{\beta}\vec{\beta}^T \end{bmatrix} \\
\Delta\mathbf{L}(-\vec{\beta}) &= \begin{bmatrix} \Delta\gamma & \Delta\gamma\vec{\beta}^T + \gamma\Delta\vec{\beta}^T \\ \Delta\gamma\vec{\beta} + \gamma\Delta\vec{\beta} & \Delta\kappa\vec{\beta}\vec{\beta}^T + \kappa\Delta\vec{\beta}\vec{\beta}^T + \kappa\vec{\beta}\Delta\vec{\beta}^T \end{bmatrix}
\end{aligned}$$

But now we need to find $\Delta\gamma$ and $\Delta\kappa$.

$$\frac{\partial\gamma}{\partial\beta_x} = \frac{\partial}{\partial\beta_x} \frac{1}{\sqrt{1-\beta_x^2-\beta_y^2-\beta_z^2}} = -\frac{1}{2}(1-\beta_x^2-\beta_y^2-\beta_z^2)^{-3/2}(-2\beta_x) = \gamma^3\beta_x$$

$$\Delta\gamma = \frac{\partial\gamma}{\partial\beta_x}\Delta\beta_x + \frac{\partial\gamma}{\partial\beta_y}\Delta\beta_y + \frac{\partial\gamma}{\partial\beta_z}\Delta\beta_z$$

$$\Delta\gamma = \gamma^3(\vec{\beta}\cdot\Delta\vec{\beta})$$

$$\kappa = \frac{\gamma^2}{\gamma+1}$$

$$\Delta\kappa = \frac{2\gamma(\gamma+1)-\gamma^2}{(\gamma+1)^2}\Delta\gamma = \frac{\gamma^2+2\gamma}{(\gamma+1)^2}\Delta\gamma = \frac{\gamma(\gamma+1)}{(\gamma+1)^2}\Delta\gamma \equiv \lambda\Delta\gamma \quad \text{where } \lambda \equiv \frac{\gamma(\gamma+1)}{(\gamma+1)^2}.$$

With that, we can write:

$$\Delta\mathbf{L}(-\vec{\beta}) = \begin{bmatrix} \Delta\gamma & \Delta\gamma\vec{\beta}^T + \gamma\Delta\vec{\beta}^T \\ \Delta\gamma\vec{\beta} + \gamma\Delta\vec{\beta} & \lambda\Delta\gamma\vec{\beta}\vec{\beta}^T + \kappa\Delta\vec{\beta}\vec{\beta}^T + \kappa\vec{\beta}\Delta\vec{\beta}^T \end{bmatrix}$$

This is (maybe) easier to write as the sum of three matrices:

$$\Delta \mathbf{L}(-\vec{\beta}) = \Delta \gamma \begin{bmatrix} 1 & \vec{\beta}^T \\ \vec{\beta} & \lambda \vec{\beta} \vec{\beta}^T \end{bmatrix} + \gamma \begin{bmatrix} 0 & \Delta \vec{\beta}^T \\ \Delta \vec{\beta} & \mathbf{0} \end{bmatrix} + \kappa \begin{bmatrix} 0 & \vec{0}^T \\ \vec{0} & \vec{\beta} \Delta \vec{\beta}^T + \Delta \vec{\beta} \vec{\beta}^T \end{bmatrix}$$

Finally, we can find the second matrix product in the expression for \mathbf{M} .

$$\begin{aligned} \Delta \mathbf{L}(-\vec{\beta}) \mathbf{L}(\vec{\beta}) &= \Delta \gamma \begin{bmatrix} 1 & \vec{\beta}^T \\ \vec{\beta} & \lambda \vec{\beta} \vec{\beta}^T \end{bmatrix} \mathbf{L}(\vec{\beta}) + \gamma \begin{bmatrix} 0 & \Delta \vec{\beta}^T \\ \Delta \vec{\beta} & \mathbf{0} \end{bmatrix} \mathbf{L}(\vec{\beta}) + \kappa \begin{bmatrix} 0 & \vec{0}^T \\ \vec{0} & \vec{\beta} \Delta \vec{\beta}^T + \Delta \vec{\beta} \vec{\beta}^T \end{bmatrix} \mathbf{L}(\vec{\beta}) \\ &= \Delta \gamma \begin{bmatrix} 1 & \vec{\beta}^T \\ \vec{\beta} & \lambda \vec{\beta} \vec{\beta}^T \end{bmatrix} \begin{bmatrix} \gamma & -\gamma \vec{\beta}^T \\ -\gamma \vec{\beta} & \mathbf{I} + \kappa \vec{\beta} \vec{\beta}^T \end{bmatrix} = \Delta \gamma \begin{bmatrix} \gamma(1-\beta^2) & -\gamma \vec{\beta}^T + \vec{\beta}^T + \kappa \beta^2 \vec{\beta}^T \\ \gamma \vec{\beta} - \gamma \lambda \beta^2 \vec{\beta} & -\gamma \vec{\beta} \vec{\beta}^T + \lambda \vec{\beta} \vec{\beta}^T + \kappa \lambda \beta^2 \vec{\beta} \vec{\beta}^T \end{bmatrix} \\ &= \Delta \gamma \begin{bmatrix} 1/\gamma & \vec{\beta}^T (-\gamma+1+\gamma-1) \\ \gamma \vec{\beta} (1-\lambda \beta^2) & \vec{\beta} \vec{\beta}^T [-\gamma+\lambda+(\gamma-1)\lambda] \end{bmatrix} \quad \text{as } \kappa \beta^2 = \gamma-1. \text{ Now use } 1-\lambda \beta^2 = \frac{2}{\gamma(\gamma+1)} \\ &= \Delta \gamma \begin{bmatrix} 1/\gamma & \vec{0}^T \\ 2\vec{\beta}/(\gamma+1) & \vec{\beta} \vec{\beta}^T [-\gamma(1-\lambda)] \end{bmatrix} = \gamma^3 (\vec{\beta} \cdot \Delta \vec{\beta}) \begin{bmatrix} 1/\gamma & \vec{0}^T \\ 2\vec{\beta}/(\gamma+1) & -\gamma \vec{\beta} \vec{\beta}^T /(\gamma+1)^2 \end{bmatrix} \quad \text{as } 1-\lambda = \frac{1}{(\gamma+1)^2} \\ &= (\vec{\beta} \cdot \Delta \vec{\beta}) \begin{bmatrix} \gamma^2 & \vec{0}^T \\ 2\gamma \kappa \vec{\beta} & -\kappa^2 \vec{\beta} \vec{\beta}^T \end{bmatrix} \\ \\ \gamma \begin{bmatrix} 0 & \Delta \vec{\beta}^T \\ \Delta \vec{\beta} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \gamma & -\gamma \vec{\beta}^T \\ -\gamma \vec{\beta} & \mathbf{I} + \kappa \vec{\beta} \vec{\beta}^T \end{bmatrix} &= \gamma \begin{bmatrix} -\gamma (\vec{\beta} \cdot \Delta \vec{\beta}) & \Delta \vec{\beta}^T + \kappa \vec{\beta}^T (\vec{\beta} \cdot \Delta \vec{\beta}) \\ +\gamma \Delta \vec{\beta} & -\gamma \Delta \vec{\beta} \vec{\beta}^T \end{bmatrix} \\ &= \begin{bmatrix} -\gamma^2 (\vec{\beta} \cdot \Delta \vec{\beta}) & -\gamma^2 \Delta \vec{\beta}^T \\ -\gamma \Delta \vec{\beta} - \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) & -\gamma^2 \Delta \vec{\beta} \vec{\beta}^T \end{bmatrix} \\ \kappa \begin{bmatrix} 0 & \vec{0}^T \\ \vec{0} & \vec{\beta} \Delta \vec{\beta}^T + \Delta \vec{\beta} \vec{\beta}^T \end{bmatrix} \begin{bmatrix} \gamma & -\gamma \vec{\beta}^T \\ -\gamma \vec{\beta} & \mathbf{I} + \kappa \vec{\beta} \vec{\beta}^T \end{bmatrix} &= \kappa \begin{bmatrix} 0 & \vec{0}^T \\ -\gamma \beta^2 \Delta \vec{\beta} - \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) \vec{\beta} & \vec{\beta} \Delta \vec{\beta}^T + \Delta \vec{\beta} \vec{\beta}^T + \kappa \beta^2 \Delta \vec{\beta} \vec{\beta}^T + \kappa (\vec{\beta} \cdot \Delta \vec{\beta}) \vec{\beta} \vec{\beta}^T \end{bmatrix} \end{aligned}$$

Now we'll add these terms together:

$$\begin{aligned}
\Delta \mathbf{L}(-\vec{\beta}) \mathbf{L}(\vec{\beta}) &= (\vec{\beta} \cdot \Delta \vec{\beta}) \begin{bmatrix} \gamma^2 & \vec{0}^T \\ 2\gamma\kappa\vec{\beta} & -\kappa^2 \vec{\beta}\vec{\beta}^T \end{bmatrix} + \begin{bmatrix} -\gamma^2(\vec{\beta} \cdot \Delta \vec{\beta}) & \gamma\Delta\vec{\beta}^T + \kappa\gamma\vec{\beta}^T(\vec{\beta} \cdot \Delta \vec{\beta}) \\ \gamma^2 \Delta \vec{\beta} & -\gamma^2 \Delta \vec{\beta} \vec{\beta}^T \end{bmatrix} \\
&+ \kappa \begin{bmatrix} 0 & \vec{0}^T \\ -\gamma\beta^2 \Delta \vec{\beta} - \gamma(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta} & \vec{\beta}\Delta\vec{\beta}^T + \Delta\vec{\beta}\vec{\beta}^T + \kappa\Delta\beta^2 \vec{\beta}\vec{\beta}^T + \kappa(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta}\vec{\beta}^T \end{bmatrix} \\
&= \begin{bmatrix} \gamma^2(\vec{\beta} \cdot \Delta \vec{\beta}) - \gamma^2(\vec{\beta} \cdot \Delta \vec{\beta}) & \gamma\Delta\vec{\beta}^T + \kappa\gamma\vec{\beta}^T(\vec{\beta} \cdot \Delta \vec{\beta}) \\ 2\gamma\kappa(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta} + \gamma^2 \Delta \vec{\beta} & -\kappa^2 \vec{\beta}\vec{\beta}^T(\vec{\beta} \cdot \Delta \vec{\beta}) - \gamma^2 \Delta \vec{\beta} \vec{\beta}^T + \kappa\vec{\beta}\Delta\vec{\beta}^T \\ -\kappa\gamma\beta^2 \Delta \vec{\beta} - \kappa\gamma(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta} & + \kappa\Delta\vec{\beta}\vec{\beta}^T + \kappa^2 \beta^2 \Delta\vec{\beta}\vec{\beta}^T + \kappa^2(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta}\vec{\beta}^T \end{bmatrix} \\
&= \begin{bmatrix} 0 & \gamma\Delta\vec{\beta}^T + \kappa\gamma(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta}^T \\ \gamma\Delta\vec{\beta} + \kappa\gamma(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta} & [-\gamma^2 + \kappa + \kappa(\gamma-1)]\Delta\vec{\beta} \vec{\beta}^T + \kappa\vec{\beta}\Delta\vec{\beta}^T \end{bmatrix} \text{ as } \kappa\beta^2 = \gamma-1 \\
&= \begin{bmatrix} 0 & \gamma\Delta\vec{\beta}^T + \kappa\gamma(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta}^T \\ \gamma\Delta\vec{\beta} + \kappa\gamma(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta} & -\kappa\Delta\vec{\beta} \vec{\beta}^T + \kappa\vec{\beta}\Delta\vec{\beta}^T \end{bmatrix} \text{ as } \kappa - \gamma = -\frac{\kappa}{\gamma}
\end{aligned}$$

Rewriting this result:

$$(C.21) \quad \Delta \mathbf{L}(-\vec{\beta}) \mathbf{L}(\vec{\beta}) = \begin{bmatrix} 0 & \gamma\Delta\vec{\beta}^T + \kappa\gamma(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta}^T \\ \gamma\Delta\vec{\beta} + \kappa\gamma(\vec{\beta} \cdot \Delta \vec{\beta})\vec{\beta} & -\kappa\Delta\vec{\beta} \vec{\beta}^T + \kappa\vec{\beta}\Delta\vec{\beta}^T \end{bmatrix}$$

For sake of completeness, I add some proofs for the side notes in the above derivation:

$$\begin{aligned}
\kappa &= \frac{\gamma^2}{\gamma+1} = \frac{\gamma-1}{\beta^2}, & \lambda &= \frac{\gamma(\gamma+2)}{(\gamma+1)^2} \\
\kappa\beta^2 &= \gamma-1 \\
\lambda\beta^2 - 1 &= \frac{\gamma(\gamma+2)}{(\gamma+1)^2} \beta^2 - 1 = \frac{\kappa(\gamma+2)}{\gamma(\gamma+1)} \beta^2 - 1 = \frac{(\gamma-1)(\gamma+2)}{\gamma(\gamma+1)} - 1 = \frac{\gamma^2 + \gamma - 2 - \gamma^2 - \gamma}{\gamma(\gamma+1)} = \frac{-2}{\gamma(\gamma+1)} \\
1 - \lambda &= 1 - \frac{\gamma(\gamma+2)}{(\gamma+1)^2} = \frac{\gamma^2 + 2\gamma + 1 - \gamma^2 - 2\gamma}{(\gamma+1)^2} = \frac{1}{(\gamma+1)^2} \\
\kappa - \gamma &= \frac{\gamma^2}{\gamma+1} - \gamma = \frac{\gamma^2 - \gamma^2 - \gamma}{\gamma+1} = \frac{-\gamma}{\gamma+1} = -\frac{\kappa}{\gamma}
\end{aligned}$$

No we go back to finding **M**.

$$\begin{aligned}
\mathbf{M} - \mathbf{I}_4 &= \Delta \mathbf{L}(-\vec{\beta}) \mathbf{L}(\vec{\beta}) + \mathbf{L}(-\vec{\beta}) \mathbf{S}_r \mathbf{L}(\vec{\beta}) = \\
&= \begin{bmatrix} 0 & \gamma \Delta \vec{\beta}^T + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) \vec{\beta}^T \\ \gamma \Delta \vec{\beta} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) & -\kappa \Delta \vec{\beta} \vec{\beta}^T + \kappa \vec{\beta} \Delta \vec{\beta}^T \end{bmatrix} + \begin{bmatrix} 0 & -\gamma \vec{C}^T \\ -\gamma \vec{C} & \kappa \vec{C} \vec{\beta}^T - \kappa \vec{\beta} \vec{C}^T \end{bmatrix} + \mathbf{S}_r \\
&= \begin{bmatrix} 0 & \gamma \Delta \vec{\beta}^T + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) \vec{\beta}^T - \gamma \vec{C}^T \\ \gamma \Delta \vec{\beta} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) - \gamma \vec{C} & -\kappa \Delta \vec{\beta} \vec{\beta}^T + \kappa \vec{\beta} \Delta \vec{\beta}^T + \kappa \vec{C} \vec{\beta}^T - \kappa \vec{\beta} \vec{C}^T \end{bmatrix} + \mathbf{S}_r \\
\mathbf{M} &= \mathbf{I}_4 + \mathbf{S}_r + \begin{bmatrix} 0 & \vec{D}^T + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) \vec{\beta}^T \\ \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) & -\kappa \vec{D} \vec{\beta}^T + \kappa \vec{\beta} \vec{D}^T \end{bmatrix} \\
&\quad \text{where } D \equiv \Delta \vec{\beta} - \vec{C} = \Delta \vec{\beta} - \vec{s} \times \vec{\beta}
\end{aligned}$$

Now let's multiply the direction four-vector by this:

$$\begin{aligned}
\mathbf{M} \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} &= \mathbf{I}_4 \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} + \mathbf{S}_r \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} + \begin{bmatrix} 0 & \vec{D}^T + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) \vec{\beta}^T \\ \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) & \kappa \vec{\beta} \vec{D}^T - \kappa \vec{D} \vec{\beta}^T \end{bmatrix} \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{s} \times \hat{u} \end{bmatrix} + \begin{bmatrix} \gamma (\vec{D} \cdot \hat{u}) + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) (\vec{\beta} \cdot \hat{u}) \\ \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) + \kappa \vec{\beta} (\vec{D} \cdot \hat{u}) - \kappa \vec{D} (\vec{\beta} \cdot \hat{u}) \end{bmatrix} \\
&= \begin{bmatrix} 1 + \gamma (\vec{D} \cdot \hat{u}) + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) (\vec{\beta} \cdot \hat{u}) \\ \hat{u} - \hat{u} \times \vec{s} + \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) + \kappa \hat{u} \times (\vec{\beta} \times \vec{D}) \end{bmatrix} = \begin{bmatrix} U \\ \vec{U} \end{bmatrix} \quad \text{using the BAC minus CAB rule}
\end{aligned}$$

$$(C.22) \quad \hat{u}_f = \frac{\vec{U}}{U} = \frac{\hat{u} - \hat{u} \times \vec{s} + \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) + \kappa \hat{u} \times (\vec{\beta} \times \vec{D})}{1 + \gamma (\vec{D} \cdot \hat{u}) + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) (\vec{\beta} \cdot \hat{u})}$$

where \hat{u}_f is the direction of the tail in the lab and \hat{u} is the direction of the head in the lab. This expression holds for any \hat{u} ; however in the special case that $\hat{u} = \vec{\beta} + \xi \Delta \vec{\beta}$, we know that $\hat{u}_f = \hat{u}$.

This gives us

$$\begin{aligned}
\hat{u} &= \frac{\hat{u} - \hat{u} \times \vec{s} + \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) + \kappa \hat{u} \times (\vec{\beta} \times \vec{D})}{1 + \gamma (\vec{D} \cdot \hat{u}) + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) (\vec{\beta} \cdot \hat{u})} \\
\hat{u} [1 + \gamma (\vec{D} \cdot \hat{u}) + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) (\vec{\beta} \cdot \hat{u})] &= \hat{u} - \hat{u} \times \vec{s} + \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) + \kappa \hat{u} \times (\vec{\beta} \times \vec{D}) \\
\gamma (\vec{D} \cdot \hat{u}) \hat{u} + \kappa \gamma (\vec{\beta} \cdot \Delta \vec{\beta}) (\vec{\beta} \cdot \hat{u}) \hat{u} &= -\hat{u} \times \vec{s} + \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) + \kappa \hat{u} \times (\vec{\beta} \times \vec{D}) \\
&= -\hat{u} \times \vec{s} + \gamma \vec{D} + \kappa \gamma \vec{\beta} (\vec{\beta} \cdot \Delta \vec{\beta}) + \kappa \vec{\beta} (\hat{u} \cdot \vec{D}) - \kappa \vec{D} (\hat{u} \cdot \vec{\beta})
\end{aligned}$$

$$\begin{aligned}
& -\hat{u} \times \bar{s} + \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) [\bar{\beta} - (\bar{\beta} \cdot \hat{u}) \hat{u}] = (\bar{D} \cdot \hat{u}) (\gamma \hat{u} - \kappa \bar{\beta}) + [\kappa (\hat{u} \cdot \bar{\beta}) - \gamma] \bar{D} \quad \text{rearranging and regrouping} \\
& = (\bar{D} \cdot \hat{u}) (\gamma \hat{u} - \kappa \bar{\beta}) - [(\gamma \hat{u} - \kappa \bar{\beta}) \cdot \hat{u}] \bar{D} \quad \hat{u} \cdot \hat{u} = 1 \\
& = \hat{u} \times [(\gamma \hat{u} - \kappa \bar{\beta}) \times \bar{D}] \quad \text{BAC minus CAB rule} \\
& \hat{u} \times [(\gamma \hat{u} - \kappa \bar{\beta}) \times \bar{D}] + \hat{u} \times \bar{s} = \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) [\bar{\beta} (\hat{u} \cdot \hat{u}) - \hat{u} (\bar{\beta} \cdot \hat{u})] \quad \hat{u} \cdot \hat{u} = 1 \\
& = \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) [\hat{u} \times (\bar{\beta} \times \hat{u})] \\
& \hat{u} \times [(\gamma \hat{u} - \kappa \bar{\beta}) \times \bar{D} + \bar{s}] = \hat{u} \times [(\bar{\beta} \times \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \hat{u})]
\end{aligned}$$

Note that on each side of the equation we now have $\hat{u} \times$ something. We can't simply equate the expressions in square brackets, though we might be tempted to do so. But what we can do is say that the expressions are the same except that we can add a constant time \hat{u} to one side or the other. We can do this because $\hat{u} \times \hat{u} = 0$. Thus, if k is an arbitrary constant:

$$\begin{aligned}
& (\gamma \hat{u} - \kappa \bar{\beta}) \times \bar{D} + \bar{s} = (\bar{\beta} \times \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \hat{u}) + k \hat{u} \quad \text{so} \\
& \hat{u} \times [(\gamma \hat{u} - \kappa \bar{\beta}) \times \bar{D} + \bar{s}] = \hat{u} \times [(\bar{\beta} \times \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \hat{u}) + k \hat{u}] \\
& \Rightarrow \hat{u} \times [(\gamma \hat{u} - \kappa \bar{\beta}) \times \bar{D} + \bar{s}] = \hat{u} \times [(\bar{\beta} \times \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \hat{u})] + k \hat{u} \times \hat{u} = \hat{u} \times [(\bar{\beta} \times \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \hat{u})]
\end{aligned}$$

Having reduced this expression this far, we now need to substitute $\hat{u} = \bar{\beta} + \xi \Delta \bar{\beta}$ and

$$\begin{aligned}
& \bar{D} = \Delta \bar{\beta} - \bar{s} \times \bar{\beta} \\
& (\gamma \hat{u} - \kappa \bar{\beta}) \times \bar{D} + \bar{s} = [\bar{\beta} \times \kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \hat{u}] + k \hat{u} \\
& (\gamma \bar{\beta} + \xi \gamma \Delta \bar{\beta} - \kappa \bar{\beta}) \times (\Delta \bar{\beta} - \bar{s} \times \bar{\beta}) + \bar{s} = [\kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \bar{\beta} \times (\bar{\beta} + \xi \Delta \bar{\beta})] + k (\bar{\beta} + \xi \Delta \bar{\beta}) \\
& [(\gamma - \kappa) \bar{\beta} + \xi \gamma \Delta \bar{\beta}] \times (\Delta \bar{\beta} - \bar{s} \times \bar{\beta}) + \bar{s} = [\kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \bar{\beta} \times (\bar{\beta} + \xi \Delta \bar{\beta})] + k (\bar{\beta} + \xi \Delta \bar{\beta}) \\
& \left(\frac{\kappa}{\gamma} \bar{\beta} + \xi \gamma \Delta \bar{\beta} \right) \times (\Delta \bar{\beta} - \bar{s} \times \bar{\beta}) + \bar{s} = [\kappa \gamma (\bar{\beta} \cdot \Delta \bar{\beta}) \bar{\beta} \times (\bar{\beta} + \xi \Delta \bar{\beta})] + k (\bar{\beta} + \xi \Delta \bar{\beta}) \quad \text{as } \kappa - \gamma = -\frac{\kappa}{\gamma} \text{ (see above)} \\
& \kappa (\bar{\beta} \times \Delta \bar{\beta}) - \kappa [\bar{\beta} \times (\bar{s} \times \bar{\beta})] - \xi \gamma^2 [\Delta \bar{\beta} \times (\bar{s} \times \bar{\beta})] + \bar{\gamma} \bar{s} \\
& = \xi \kappa \gamma^2 (\bar{\beta} \cdot \Delta \bar{\beta}) (\bar{\beta} \times \Delta \bar{\beta}) + k \gamma (\bar{\beta} + \xi \Delta \bar{\beta}) \quad \text{as } \bar{A} \times \bar{A} = 0 \\
& \kappa (\bar{\beta} \times \Delta \bar{\beta}) - \kappa [\bar{s} \beta^2 - \bar{\beta} (\bar{\beta} \cdot \bar{s})] - \xi \gamma^2 [\bar{s} (\Delta \bar{\beta} \cdot \bar{\beta}) - \bar{\beta} (\Delta \bar{\beta} \cdot \bar{s})] + \bar{\gamma} \bar{s} \\
& = \xi \kappa \gamma^2 (\bar{\beta} \cdot \Delta \bar{\beta}) (\bar{\beta} \times \Delta \bar{\beta}) + k \gamma (\bar{\beta} + \xi \Delta \bar{\beta}) \quad \text{BAC minus CAB rule} \\
& -\bar{s} [\kappa \beta^2 + \xi \gamma^2 (\Delta \bar{\beta} \cdot \bar{\beta}) - \gamma] = \kappa (\bar{\beta} \times \Delta \bar{\beta}) [\xi \gamma^2 (\bar{\beta} \cdot \Delta \bar{\beta}) - 1] - \kappa \bar{\beta} (\bar{\beta} \cdot \bar{s}) - \xi \gamma^2 \bar{\beta} (\Delta \bar{\beta} \cdot \bar{s}) + k \gamma (\bar{\beta} + \xi \Delta \bar{\beta}) \\
& -\bar{s} [-1 + \xi \gamma^2 (\Delta \bar{\beta} \cdot \bar{\beta})] = \kappa (\bar{\beta} \times \Delta \bar{\beta}) [\xi \gamma^2 (\bar{\beta} \cdot \Delta \bar{\beta}) - 1] - \kappa \bar{\beta} (\bar{\beta} \cdot \bar{s}) - \xi \gamma^2 \bar{\beta} (\Delta \bar{\beta} \cdot \bar{s}) + k \gamma (\bar{\beta} + \xi \Delta \bar{\beta}) \\
& \bar{s} [-1 + \xi \gamma^2 (\Delta \bar{\beta} \cdot \bar{\beta})] = \kappa (\Delta \bar{\beta} \times \bar{\beta}) [\xi \gamma^2 (\bar{\beta} \cdot \Delta \bar{\beta}) - 1] + \kappa \bar{\beta} (\bar{\beta} \cdot \bar{s}) + \xi \gamma^2 \bar{\beta} (\Delta \bar{\beta} \cdot \bar{s}) - k \gamma (\bar{\beta} + \xi \Delta \bar{\beta})
\end{aligned}$$

At this point we note that the coefficients of \bar{s} on the left and $\kappa (\Delta \bar{\beta} \times \bar{\beta})$ on right are identical. If the other three terms on the right were not there, we easily solve for \bar{s} to give:

$$(C.23) \quad \bar{s} = \kappa (\Delta \bar{\beta} \times \bar{\beta}).$$

If that were true, then the terms of the right would be:

$$\begin{aligned}\kappa\vec{\beta}(\vec{\beta}\cdot\vec{s}) &= \kappa^2\vec{\beta}[\vec{\beta}\cdot(\Delta\vec{\beta}\times\vec{\beta})] = 0 && \text{by the dot - cross rule} \\ \xi\gamma^2\vec{\beta}(\Delta\vec{\beta}\cdot\vec{s}) &= \xi\kappa\gamma^2\vec{\beta}[\Delta\vec{\beta}\cdot(\Delta\vec{\beta}\times\vec{\beta})] = 0 && \text{by the dot - cross rule} \\ -k\gamma(\vec{\beta} + \xi\Delta\vec{\beta}) &= 0 \text{ if and only if } k = 0\end{aligned}$$

Since we are at liberty to set k equal to zero, we have found \vec{s} and hence the rest-frame rotation matrix \mathbf{R}_r .

Now that we have found \mathbf{R}_r , we can put this expression in Eq. (C.22) with an arbitrary value of the head direction \hat{u} to find the tail direction \hat{u}_f . But first, let us find an expression for \vec{D} :

$$\begin{aligned}\vec{D} &= \Delta\vec{\beta} - \vec{s}\times\vec{\beta} = \Delta\vec{\beta} - \kappa(\Delta\vec{\beta}\times\vec{\beta})\times\vec{\beta} = \Delta\vec{\beta} + \kappa\vec{\beta}\times(\Delta\vec{\beta}\times\vec{\beta}) \\ &= \Delta\vec{\beta} + \kappa\Delta\vec{\beta}\beta^2 - \kappa\vec{\beta}(\vec{\beta}\cdot\Delta\vec{\beta}) && \text{BAC minus CAB rule} \\ &= \Delta\vec{\beta} + (\gamma-1)\Delta\vec{\beta} - \kappa\vec{\beta}(\vec{\beta}\cdot\Delta\vec{\beta}) && \text{as } \kappa\beta^2 = \lambda - 1 \\ &= \gamma\Delta\vec{\beta} - \kappa\vec{\beta}(\vec{\beta}\cdot\Delta\vec{\beta})\end{aligned}$$

$$\begin{aligned}\hat{u}_f &= \frac{\hat{u} - \hat{u}\times\vec{s} + \gamma\vec{D} + \kappa\gamma\vec{\beta}(\vec{\beta}\cdot\Delta\vec{\beta}) + \kappa\hat{u}\times(\vec{\beta}\times\vec{D})}{1 + \gamma(\vec{D}\cdot\hat{u}) + \kappa\gamma(\vec{\beta}\cdot\Delta\vec{\beta})(\vec{\beta}\cdot\hat{u})} \\ \hat{u}_f &= \frac{\hat{u} - \hat{u}\times[\kappa(\Delta\vec{\beta}\times\vec{\beta})] + \gamma^2\Delta\vec{\beta} - \kappa\gamma\vec{\beta}(\vec{\beta}\cdot\Delta\vec{\beta}) + \kappa\gamma\vec{\beta}(\vec{\beta}\cdot\Delta\vec{\beta}) + \kappa\hat{u}\times(\vec{\beta}\times\vec{D})}{1 + \gamma^2(\Delta\vec{\beta}\cdot\hat{u}) - \kappa\gamma(\vec{\beta}\cdot\hat{u})(\vec{\beta}\cdot\Delta\vec{\beta}) + \kappa\gamma(\vec{\beta}\cdot\Delta\vec{\beta})(\vec{\beta}\cdot\hat{u})} \\ &= \frac{\hat{u} - \hat{u}\times[\kappa(\Delta\vec{\beta}\times\vec{\beta})] + \gamma^2\Delta\vec{\beta} + \kappa\hat{u}\times(\vec{\beta}\times\vec{D})}{1 + \gamma^2(\Delta\vec{\beta}\cdot\hat{u})} \\ &= \frac{\hat{u} + \hat{u}\times\kappa(\vec{\beta}\times\Delta\vec{\beta}) + \gamma^2\Delta\vec{\beta} + \kappa\hat{u}\times(\vec{\beta}\times\vec{D})}{1 + \gamma^2(\Delta\vec{\beta}\cdot\hat{u})} \\ &= \frac{\hat{u} + \kappa\hat{u}\times[\vec{\beta}\times(\Delta\vec{\beta} + \gamma\Delta\vec{\beta} - \kappa\vec{\beta}(\vec{\beta}\cdot\Delta\vec{\beta}))] + \gamma^2\Delta\vec{\beta}}{1 + \gamma^2(\Delta\vec{\beta}\cdot\hat{u})} \\ &= \frac{\hat{u} + \kappa\hat{u}\times[\vec{\beta}\times(\gamma+1)\Delta\vec{\beta}] + \gamma^2\Delta\vec{\beta}}{1 + \gamma^2(\Delta\vec{\beta}\cdot\hat{u})} \quad \text{as } \vec{\beta}\times\vec{\beta} = 0 \\ &= \frac{\hat{u} + \gamma^2\hat{u}\times(\vec{\beta}\times\Delta\vec{\beta}) + \gamma^2\Delta\vec{\beta}}{1 + \gamma^2(\Delta\vec{\beta}\cdot\hat{u})} \quad \text{as } \kappa(\gamma+1) = \gamma^2 \\ &= \frac{\hat{u} + \gamma^2[\Delta\vec{\beta} + \hat{u}\times(\vec{\beta}\times\Delta\vec{\beta})]}{1 + \gamma^2(\hat{u}\cdot\Delta\vec{\beta})}\end{aligned}$$

We have really solved the problem at this point, as we know what the tail direction is in terms of the head direction; however, we still would like to find the lab rotation matrix. We know:

$$(\mathbf{I}_4 + \mathbf{S}_l) \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = \begin{bmatrix} 1 \\ \hat{u}_f \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} + \mathbf{S}_l \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = \begin{bmatrix} 1 \\ \hat{u}_f \end{bmatrix}$$

$$\mathbf{S}_l \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = \begin{bmatrix} 1 \\ \hat{u}_f \end{bmatrix} - \begin{bmatrix} 1 \\ \hat{u} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{u}_f - \hat{u} \end{bmatrix}$$

$$\begin{aligned} \hat{u}_f - \hat{u} &= \frac{\hat{u} + \gamma^2 [\Delta \vec{\beta} + \hat{u} \times (\vec{\beta} \times \Delta \vec{\beta})]}{1 + \gamma^2 (\hat{u} \cdot \Delta \vec{\beta})} - \hat{u} \\ &= \frac{\hat{u} + \gamma^2 [\Delta \vec{\beta} + \hat{u} \times (\vec{\beta} \times \Delta \vec{\beta})] - \hat{u} - \gamma^2 (\hat{u} \cdot \Delta \vec{\beta}) \hat{u}}{1 + \gamma^2 (\hat{u} \cdot \Delta \vec{\beta})} \\ &= \frac{\gamma^2 \hat{u} \times (\vec{\beta} \times \Delta \vec{\beta}) + \gamma^2 \Delta \vec{\beta} - \gamma^2 (\hat{u} \cdot \Delta \vec{\beta}) \hat{u}}{1 + \gamma^2 (\hat{u} \cdot \Delta \vec{\beta})} \\ &= \frac{\gamma^2 \hat{u} \times (\vec{\beta} \times \Delta \vec{\beta}) + \gamma^2 \hat{u} \times (\Delta \vec{\beta} \times \hat{u})}{1 + \gamma^2 (\hat{u} \cdot \Delta \vec{\beta})} \quad \text{BAC minus CAB} \\ &= \frac{\gamma^2 \hat{u} \times [\Delta \vec{\beta} \times (\hat{u} - \vec{\beta})]}{1 + \gamma^2 (\hat{u} \cdot \Delta \vec{\beta})} \end{aligned}$$

At this point we can make use of the relation $(1 + \Delta x)^n \approx 1 + n \Delta x$, a Taylor series expansion valid for small Δx , to simplify this expression.

$$\begin{aligned} \hat{u}_f - \hat{u} &= \frac{\gamma^2 \hat{u} \times [\Delta \vec{\beta} \times (\hat{u} - \vec{\beta})]}{1 + \gamma^2 (\hat{u} \cdot \Delta \vec{\beta})} \\ &\approx \gamma^2 \hat{u} \times [\Delta \vec{\beta} \times (\hat{u} - \vec{\beta})] [1 - \gamma^2 (\hat{u} \cdot \Delta \vec{\beta})] \\ &\approx \gamma^2 \hat{u} \times [\Delta \vec{\beta} \times (\hat{u} - \vec{\beta})] \end{aligned}$$

As usual, we drop terms that are quadratic in infinitesimals.

We also know that this can be expressed in terms of a rotation as:

$$\begin{aligned} \hat{u}_f - \hat{u} &= \gamma^2 \hat{u} \times [\Delta \vec{\beta} \times (\hat{u} - \vec{\beta})] = \vec{s}_l \times \hat{u} \\ \Rightarrow \vec{s}_l &= \gamma^2 (\hat{u} - \vec{\beta}) \times \Delta \vec{\beta} \end{aligned} \tag{C.24}$$

As we described earlier, the magnitude of this vector is the rotation angle and its direction is along the rotation axis.

Now we need to find the acceleration thread in terms of the rotation angle. From Fig. C.3, we see that the length of the acceleration thread is just

$$\ell_a = r_t \Delta\theta \approx r_h \Delta\theta .$$

The length of the head and tail lines are essentially the same as we let the thread length go to zero. That is, $r_t \Delta\theta = (r_h - \Delta r) \Delta\theta = r_h \Delta\theta$ if we drop the term that is quadratic in infinitesimals. The magnitude of the rotation angle is the magnitude of the \vec{s} vector:

$$\Delta\theta = s.$$

Now we want to find the direction of the acceleration thread. First we see from Fig. C.3 that the distance from point T to the head of the acceleration thread is the same as the distance from T to the tail of the thread. In the limit of the thread becoming very short, the direction of the thread becomes perpendicular to the direction of the head line. Since $\vec{s}_l = \gamma^2 (\hat{u} - \vec{\beta}) \times \Delta\vec{\beta}$ and $\vec{r}_r = r_h (\hat{u} - \vec{\beta})$, we see that \vec{s} is in the direction of $\vec{r}_r \times \vec{\alpha}$. In Fig. C.3, the direction of \vec{s} is into the page. The direction of the acceleration thread is then perpendicular to \vec{s} and perpendicular to the head line as well. This suggests a cross product is operative here. By applying the right-hand rule, we see that the thread direction is $\hat{r}_h \times \hat{s}$. Combining the expressions for the thread length and direction, we have:

$$\begin{aligned} \vec{\ell}_a &= \vec{r}_h \times \vec{s}_l \\ &= \gamma^2 \vec{r}_h \times [(\hat{u} - \vec{\beta}) \times \Delta\vec{\beta}] \\ &= \gamma^2 \hat{r}_h \times [r_h (\hat{u} - \vec{\beta}) \times \Delta\vec{\beta}] \\ &= \gamma^2 \hat{r}_h \times [(\vec{r}_h - r_h \vec{\beta}) \times \Delta\vec{\beta}] \\ &= \gamma^2 \hat{r}_h \times (\vec{r}_r \times \Delta\vec{\beta}) \\ &= \gamma^2 \hat{r}_h \times (\vec{r}_r \times \vec{\alpha} \gamma \ell_0) \\ \vec{\ell}_a &= \gamma^3 \ell_0 \hat{r}_h \times (\vec{r}_r \times \vec{\alpha}) \end{aligned} \tag{C.25}$$

We are finally in a position to calculate the electric field of the accelerating point charge. Using Eqs. (2.2) and (2.5) :

$$\begin{aligned}
\vec{E}_a &= \frac{e}{\epsilon_0} \vec{\ell} \nu \quad \gamma^3 \ell_0 \hat{r}_h \times (\vec{r}_r \times \vec{\alpha}) \quad \nu = \frac{N_0 q_s r_h}{4\pi e c r_0^3} \\
\vec{E}_a &= \frac{e}{\epsilon_0} \gamma^3 \ell_0 \hat{r}_h \times (\vec{r}_r \times \vec{\alpha}) \frac{N_0 q_s r_h}{4\pi e c r_0^3} \\
&= \frac{q_s}{4\pi \epsilon_0} \frac{N_0 \ell_0}{c} \frac{\vec{r}_h \times (\vec{r}_r \times \vec{\alpha})}{(r_0 / \gamma)^3} \quad \text{using } \vec{r}_h = r_h \hat{r}_h \\
&= \frac{q_s}{4\pi \epsilon_0} \frac{\vec{r}_h \times (\vec{r}_r \times \vec{\alpha})}{\rho^3} \quad \text{using } N_0 \ell_0 = c \text{ and } r_0 = \gamma \rho
\end{aligned}$$

Rewriting this, we have:

$$(C.26) \quad \vec{E}_a = \frac{q_s}{4\pi \epsilon_0} \frac{\vec{r}_h \times (\vec{r}_r \times \vec{\alpha})}{\rho^3}$$

The total electric field is the sum of the constant velocity field we found earlier and the acceleration field. But what about the magnetic field produced by the accelerating charge? In the constant velocity case we found in Sect. C.5 that the magnetic field was related to the electric field by the expression:

$$\vec{B}_v = \frac{1}{c} \vec{\beta}_s \times \vec{E}_v = \frac{1}{c} \hat{r}_h \times \vec{E}_v.$$

But our derivation of these formulas assumed that the electric field was in the direction of the ray line. With the acceleration electric field in a completely different direction, we have to rework the problem. The method closely parallels that of Sect. 3.5:

1. We know the acceleration electric field from Eq. (C.26). If a field particle in the lab frame is at rest, the acceleration causes a force of $\vec{F}_a = q_f \vec{E}_a$ on the field particle.
2. We find the force four-vector in the lab frame and use a Lorentz transformation to find the force in the rest frame of the source.
3. We use Eq. (B.17) to find the electric field in the rest frame of the source frame and Eq. (B.12) to find the acceleration of the source in this frame. We then calculate the force due to the electric field in the rest frame. The force due to the magnetic field is the difference between this force and the total force.
4. Knowing that the force of the magnetic field is $q_f \vec{v}_f \times \vec{B} = q_f \vec{\beta}_f \times c\vec{B}$, we have some information about the magnetic field.
5. We require the electric and magnetic fields to be perpendicular to obtain the rest of the information necessary to determine the magnetic field completely.
6. We use Eq. (B.17) to transform the magnetic field back to the lab frame.

The algebra here is much more complicated than that of Sect. 3.5, but the result is similar.

$$\vec{B}_a = \frac{1}{c} \hat{r}_h \times \vec{E}_a.$$

Note that the expression $\vec{B} = \frac{1}{c} \vec{\beta}_s \times \vec{E}$ is valid for velocity fields, but not for acceleration fields.

This then leads to the completely general result for the electric and magnetic fields of a point particle:

General Equations for Fields

$$(C.27) \quad \vec{E} = \vec{E}_v + \vec{E}_a = \frac{q_s}{4\pi\epsilon_0} \frac{\vec{r}_r}{\gamma_s^2 \rho^3} + \frac{q_s}{4\pi\epsilon_0} \frac{\vec{r}_h \times (\vec{r}_r \times \vec{\alpha}_s)}{\rho^3}$$

$$\vec{B} = \frac{1}{c} \hat{r}_h \times \vec{E}$$

where

$$\rho = r_r \sqrt{1 - \beta_s^2 \sin^2 \psi}$$

C.8. Exact Treatment of the Fields of a Wire with Constant Current

Let's begin by taking a wire at time $ct=0$. We take the net charge on any section of the wire to be zero, ignoring a charge gradient that naturally occurs on the surface of a wire with resistance. We call the linear density of fixed negative charges $-\lambda$ so that the density of positive charge carriers is $+\lambda$. Let's take the wire to go along the x axis of a coordinate system. Taking a slice of wire at an arbitrary x , we have the situation depicted in Fig.C.5. (It is important that we slice the wire at $ct=0$ and then determine the location of each slice when the threads were emitted. If we sliced the wire when the threads were emitted, the slices would be taken at many different times and we would have to use great care that the slices didn't overlap.)

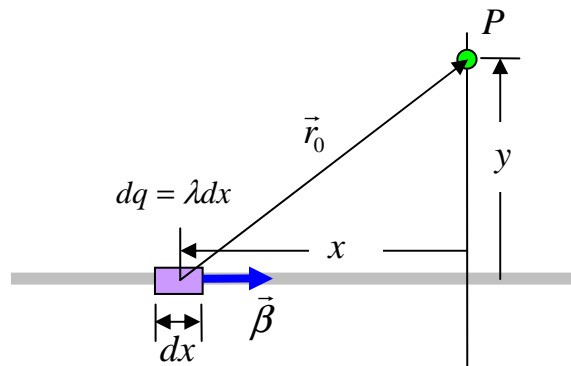


Figure C.5 Geometry for a slice of the wire.

We assume that the velocity of the charge is constant in time:

The slice in Fig. 1 is taken at time $ct=0$, but to calculate the fields, we need to know where the moving charge was located when the threads arriving at P at $ct=0$ were first emitted. As before, we call this point S . Furthermore, we need to know where the source is located at time $ct=0$. This is point U .

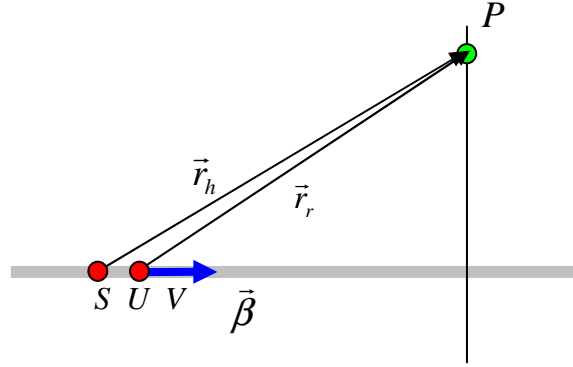


Figure C.6 Successive positions of the source.

Note that the time the threads leave point S is just $ct = -r_h$. Also, let the vectors \vec{S} and \vec{U} be the vectors from the origin to points S and U respectively. Now we can write expressions for all the vectors in the system. We want to put everything in terms of x and y .

$$\begin{aligned}\vec{U} &= \vec{S} + \vec{\beta} r_h = x\hat{x} \\ \vec{P} &= y\hat{y} \\ \vec{r}_r &= \vec{P} - \vec{U} = y\hat{y} - x\hat{x} \\ \vec{r}_h &= \vec{P} - \vec{S} = \vec{\beta} r_h + \vec{r}_r \\ &= y\hat{y} + (\beta r_h - x)\hat{x}\end{aligned}$$

We can find the magnitude of \vec{r}_h :

$$\begin{aligned}
r_h^2 &= y^2 + (\beta r_h - x)^2 \\
&= y^2 + x^2 - 2\beta x r_h + \beta^2 r_h^2 \\
r_h^2(1 - \beta^2) + 2\beta x r_h - r_r^2 &= 0 \\
\frac{r_h^2}{\gamma^2} + 2\beta x r_h - r_r^2 &= 0 \\
r_h^2 + 2\beta\gamma^2 x r_h - \gamma^2 r_r^2 &= 0 \\
r_h &= \frac{-2\beta\gamma^2 x \pm \sqrt{4\beta^2\gamma^4 x^2 + 4\gamma^2 r_r^2}}{2} \\
&= -\beta\gamma^2 x \pm \sqrt{\beta^2\gamma^4 x^2 + \gamma^2 r_r^2} \\
\text{If } \beta &= 0, \quad r_h = r_r \\
\Rightarrow r_h &= -\beta\gamma^2 x + \gamma\sqrt{\beta^2\gamma^2 x^2 + r_r^2} \\
&= -\beta\gamma^2 x + \gamma\sqrt{\beta^2\gamma^2 x^2 + x^2 + y^2} \\
&= -\beta\gamma^2 x + \gamma\sqrt{(\beta^2\gamma^2 + 1)x^2 + y^2} \\
&= -\beta\gamma^2 x + \gamma\sqrt{\gamma^2 x^2 + y^2}
\end{aligned}$$

Now we find an expression for ρ :

$$\begin{aligned}
\rho &= \frac{\vec{r}_h \cdot \vec{r}_r}{r_h r_r} = \frac{[y\hat{y} + (\beta r_h - x)\hat{x}] \cdot [y\hat{y} - x\hat{x}]}{r_h r_r} \\
&= \frac{y^2 + x^2 - \beta r_h x}{r_h} \\
&= \frac{r_h^2 + \beta r_h x - \beta^2 r_h^2}{r_h} \\
&= \frac{r_h^2(1 - \beta^2) + \beta r_h x}{r_h} \\
&= \frac{r_h}{\gamma^2} + \beta x \\
&= \frac{r_h + \beta x \gamma^2}{\gamma^2} \\
&= \frac{\gamma\sqrt{\gamma^2 x^2 + y^2}}{\gamma^2} \quad \text{using the expression for } r_h \text{ above} \\
&= \frac{\sqrt{\gamma^2 x^2 + y^2}}{\gamma}
\end{aligned}$$

The electric velocity field of the moving charge is then:

$$d\vec{E}_m = \frac{k\lambda dx \vec{r}_r}{\gamma^2 \rho^3} = \frac{\gamma k\lambda dx (y\hat{y} - x\hat{x})}{(\gamma^2 x^2 + y^2)^{3/2}}$$

The sum of the fields of the fixed charges and the moving charges in the slice is:

$$d\vec{E}_m + d\vec{E}_f = \frac{\gamma k\lambda dx (y\hat{y} - x\hat{x})}{(\gamma^2 x^2 + y^2)^{3/2}} - \frac{k\lambda dx (y\hat{y} - x\hat{x})}{(x^2 + y^2)^{3/2}}$$

We need to integrate along the entire x axis.:

$$\int_{-\infty}^{+\infty} \frac{\gamma x}{(\gamma^2 x^2 + y^2)^{3/2}} dx = \int_{-\infty}^{+\infty} \frac{x}{(x^2 + y^2)^{3/2}} dx = 0$$

$$\int_{-\infty}^{+\infty} \frac{\gamma}{(\gamma^2 x^2 + y^2)^{3/2}} dx = \frac{2}{y^2}$$

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2 + y^2)^{3/2}} dx = \frac{2}{y^2}$$

Thus we see that everything reduces to zero. Since we made no approximations in this derivation, it tells us that the electric field of a constant-velocity particle beam is identical to that of a stationary charge distribution having the same (laboratory) charge density.

C.9. Fields of a Wire with Increasing Current

Let's begin by taking a wire at time $ct=0$. We take the net charge on any section of the wire to be zero, ignoring a charge gradient that naturally occurs on the surface of a wire with resistance. We call the linear density of fixed negative charges $-\lambda$ so that the density of positive charge carriers is $+\lambda$. Let's take the wire to go along the x axis of a coordinate system. Taking a slice of wire at an arbitrary x , we have the situation depicted in Fig. C.7.

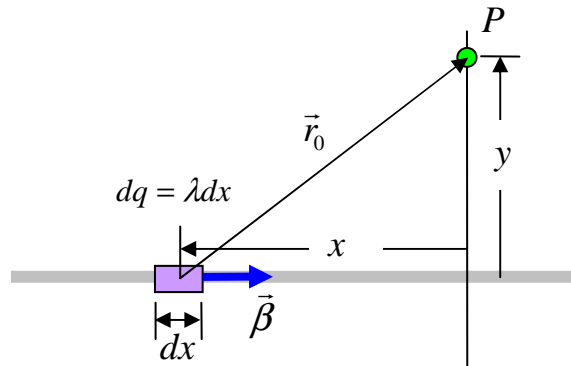


Figure C.7 Geometry for a slice of the wire.

We assume that the charge is uniformly accelerating in the $+\hat{x}$ direction. We may then write the source velocity as a function of time as

$$\vec{\beta}(ct) = \vec{\beta}_0 + \vec{\alpha} ct.$$

The slice in Fig. C.7 is taken at time $ct=0$, but to calculate the fields, we need to know where the moving charge was located when the threads arriving at P at $ct=0$ were first emitted. As before, we call this point S . Furthermore, we need to know where the source would have been at time $ct=0$ if there had been no acceleration. This is point U . Finally, we let V be the actual position of the source at $ct=0$.

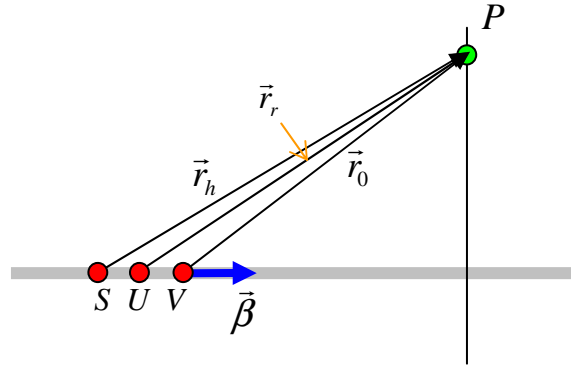


Figure C.8. Successive positions of the source.

Note that the time the threads leave point S is just $ct = -r_h$. Also, let the vectors \vec{S} , \vec{U} , and \vec{V} be the vectors from the origin to points S , U , and V respectively. Now we can write expressions for all the vectors in the system. We want to put everything in terms of x and y .

$$\begin{aligned}\vec{V} &= x\hat{x} \\ \vec{P} &= y\hat{y} \\ \vec{V} &= \vec{S} + \vec{\beta}(-r_h)r_h + \frac{1}{2}\vec{\alpha}r_h^2 \\ &= \vec{S} + \vec{\beta}_0 r_h - \frac{1}{2}\vec{\alpha}r_h^2 \\ \vec{U} &= \vec{S} + \vec{\beta}(-r_h)r_h \\ &= \vec{S} + \vec{\beta}_0 r_h - \vec{\alpha}r_h^2 \\ \\ \vec{r}_h &= \vec{P} - \vec{S} \equiv \vec{r}_0 - \Delta\vec{r}_h \\ \vec{r}_r &= \vec{P} - \vec{U} \equiv \vec{r}_0 - \Delta\vec{r}_r \\ \vec{r}_0 &= \vec{P} - \vec{V}\end{aligned}$$

$$\begin{aligned}\Delta\vec{r}_h &= \vec{r}_0 - \vec{r}_h = \vec{S} - \vec{V} = -\vec{\beta}_0 r_h + \frac{1}{2}\vec{\alpha} r_h^2 \\ \Delta\vec{r}_r &= \vec{r}_0 - \vec{r}_r = \vec{U} - \vec{V} = -\frac{1}{2}\vec{\alpha} r_h^2\end{aligned}$$

Since the drift velocity is very small, we know that \vec{r}_h and \vec{r}_0 are very nearly equal. Let's write:

$$\begin{aligned}\vec{r}_h &\equiv \vec{r}_0 - \Delta\vec{r}_h \\ \Rightarrow r_h^2 &\cong r_0^2 - 2\vec{r}_0 \cdot \Delta\vec{r}_h \\ r_h &\cong r_0 \left(1 - \frac{\hat{r}_0 \cdot \Delta\vec{r}_h}{r_0} \right) \\ &= r_0 - \hat{r}_0 \cdot \Delta\vec{r}_h\end{aligned}$$

Similarly,

$$r_r \cong r_0 - \hat{r}_0 \cdot \Delta\vec{r}_r.$$

$$\begin{aligned}\rho &= \frac{\vec{r}_h \cdot \vec{r}_r}{r_h} \cong \frac{(r_0 - \hat{r}_0 \cdot \Delta\vec{r}_h) \cdot (r_0 - \hat{r}_0 \cdot \Delta\vec{r}_r)}{r_0 \left(1 - \frac{\hat{r}_0 \cdot \Delta\vec{r}_h}{r_0} \right)} \\ &\cong (r_0^2 - \vec{r}_0 \cdot \Delta\vec{r}_h - \vec{r}_0 \cdot \Delta\vec{r}_r) \frac{1}{r_0} \left(1 + \frac{\hat{r}_0 \cdot \Delta\vec{r}_h}{r_0} \right) \\ &= (r_0 - \hat{r}_0 \cdot \Delta\vec{r}_h - \hat{r}_0 \cdot \Delta\vec{r}_r) \left(1 + \frac{\hat{r}_0 \cdot \Delta\vec{r}_h}{r_0} \right) \\ &= r_0 + \hat{r}_0 \cdot \Delta\vec{r}_h - \hat{r}_0 \cdot \Delta\vec{r}_h - \hat{r}_0 \cdot \Delta\vec{r}_r \\ &= r_0 - \hat{r}_0 \cdot \Delta\vec{r}_r \\ &= r_r\end{aligned}$$

Before we evaluate the electric field of the moving charges, we need to find one more quantity:

$$\frac{1}{\gamma^2} = 1 - \beta^2 \cong 1.$$

The electric velocity field of the moving charge is then:

$$\begin{aligned}
d\vec{E}_m &= \frac{k\lambda dx \vec{r}_r}{\gamma^2 \rho^3} = \frac{k\lambda dx (\vec{r}_0 - \Delta\vec{r}_r)}{r_0^3 \left(1 - \frac{\hat{r}_0 \cdot \Delta\vec{r}_r}{r_0}\right)^3} \\
&\cong \frac{k\lambda dx}{r_0^3} (\vec{r}_0 - \Delta\vec{r}_r) \left(1 + 3 \frac{\hat{r}_0 \cdot \Delta\vec{r}_r}{r_0}\right) \\
&\cong \frac{k\lambda dx \vec{r}_0}{r_0^3} \left(1 + 3 \frac{\hat{r}_0 \cdot \Delta\vec{r}_r}{r_0}\right) - \frac{k\lambda dx \Delta\vec{r}_r}{r_0^3}
\end{aligned}$$

The sum of the fields of the fixed charges and the moving charges in the slice is:

$$\begin{aligned}
d\vec{E}_f + d\vec{E}_m &= -\frac{k\lambda dx \vec{r}_0}{r_0^3} + \frac{k\lambda dx \vec{r}_r}{\gamma^2 \rho^3} \\
&\cong \frac{k\lambda dx \vec{r}_0}{r_0^3} \left(3 \frac{\hat{r}_0 \cdot \Delta\vec{r}_r}{r_0}\right) - \frac{k\lambda dx \Delta\vec{r}_r}{r_0^3} \\
&= \frac{3k\lambda dx \vec{r}_0}{r_0^5} (\vec{r}_0 \cdot \Delta\vec{r}_r) - \frac{k\lambda dx \Delta\vec{r}_r}{r_0^3}
\end{aligned}$$

We also note that:

$$\begin{aligned}
\vec{r}_0 &= y\hat{y} - x\hat{x} \text{ and} \\
\Delta\vec{r}_r &= -\frac{1}{2}\alpha r_h^2 \\
&\cong -\frac{1}{2}\alpha (r_0^2 - 2\vec{r}_0 \cdot \Delta\vec{r}_h) \\
&\cong -\frac{1}{2}\alpha r_0^2 \text{ as } \alpha \text{ is very small.}
\end{aligned}$$

This then leads to:

$$\begin{aligned}
d\vec{E} &= \frac{3k\lambda dx \vec{r}_0}{r_0^5} (\vec{r}_0 \cdot \Delta\vec{r}_r) - \frac{k\lambda dx \Delta\vec{r}_r}{r_0^3} \\
&= \frac{3k\lambda dx (y\hat{y} - x\hat{x})}{r_0^5} \left(\frac{1}{2}\alpha r_0^2 x\right) + \frac{k\lambda dx}{r_0^3} \left(\frac{1}{2}\alpha r_0^2 \hat{x}\right) \\
&= \frac{k\lambda \alpha dx}{2r_0^3} (3xy\hat{y} + r_0^2 \hat{x} - 3x^2 \hat{x}) \\
&= \frac{k\lambda \alpha dx}{2(x^2 + y^2)^{3/2}} [3xy\hat{y} + (y^2 - 2x^2)\hat{x}]
\end{aligned}$$

To simplify this, we need to integrate along the entire x axis. We see that the term involving \hat{y} integrates to zero as the integrand is an odd function of x evaluated over symmetric limits ($x = -\infty$ to $+\infty$).

$$\vec{E} = \hat{x} \frac{k\lambda\alpha}{2} \int_{-\infty}^{+\infty} \frac{y^2 - 2x^2}{(x^2 + y^2)^{3/2}} dx$$

We can let Maple evaluate the integral for us:

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{y^2}{(x^2 + y^2)^{3/2}} dx &= 2 \\ -2 \int_{-\infty}^{+\infty} \frac{x^2}{(x^2 + y^2)^{3/2}} dx &\rightarrow -\infty \end{aligned}$$

That result doesn't look too promising. However, we do need to remember that real wires aren't infinitely long. Let's then assume that the wire goes only from $-L$ to $+L$. Then we can evaluate the second integral:

$$\int_{-L}^{+L} \frac{y^2 - 2x^2}{(x^2 + y^2)^{3/2}} dx = \frac{6L}{\sqrt{L^2 + y^2}} - 4 \ln \left(\frac{L + \sqrt{L^2 + y^2}}{y} \right).$$

One thing we note from this integral is that when $L \gg y$, the integral becomes:

$$\text{if } L \gg y, \int_{-L}^{+L} \frac{y^2 - 2x^2}{(x^2 + y^2)^{3/2}} dx \rightarrow 6 - 4 \ln \left(\frac{2L}{y} \right) < 0.$$

This tells us that if the acceleration is in the $+\hat{x}$ direction, the field is in the $-\hat{x}$ direction, and vice-versa. Also we see that as y increases, the magnitude of the field decreases logarithmically. Since logarithms change very slowly, we see that the electric field drops off slowly as we move away from the wire.

Now let's place a square loop of wire near the current-carrying wire. We let the near edge of the loop to be at a distance y_1 from the wire and the far edge of the loop to be at a distance y_2 from the wire. Also, we let the width of the loop to be w . This is shown in Fig. C.9.

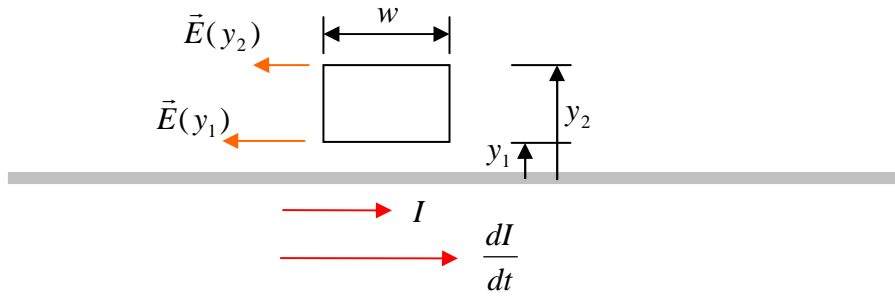


Figure C.9 Placing a wire loop near a wire with increasing current.

Positive charge carriers in the loop will be pushed to the left by the electric field of the wire. However, the charges at the bottom of the wire are pushed with a greater force than those at the top. The net result is that current will spontaneously flow around the loop in a clockwise direction. The net voltage difference in going around a complete loop is:

$$\Delta V = \oint E dl = \vec{E}(y_2)w - \vec{E}(y_1)w$$

Previously, we always concluded that the voltage around a closed loop was zero. We see that this conclusion, while valid with electrostatics, is not valid when charges accelerate. The fact that we can induce a current to flow in a wire is a very important result that we will discuss in great detail in Lesson 11.

As it turns out, we can actually calculate this voltage difference in the limit that L becomes infinite.

$$\begin{aligned} \Delta V &= E(y_2)w - E(y_1)w \\ &= \frac{k\lambda\alpha}{2} w \lim_{L \rightarrow \infty} \left\{ \frac{6L}{\sqrt{L^2 + y_2^2}} - \frac{6L}{\sqrt{L^2 + y_1^2}} - 4 \ln \left(\frac{L + \sqrt{L^2 + y_2^2}}{y_2} \right) + 4 \ln \left(\frac{L + \sqrt{L^2 + y_1^2}}{y_1} \right) \right\} \\ &= \frac{k\lambda\alpha}{2} w \lim_{L \rightarrow \infty} \left\{ \frac{6}{\sqrt{1 + \frac{y_2^2}{L^2}}} - \frac{6}{\sqrt{1 + \frac{y_1^2}{L^2}}} + 4 \ln \left(\frac{L + \sqrt{L^2 + y_1^2}}{y_1} \cdot \frac{y_2}{L + \sqrt{L^2 + y_2^2}} \right) \right\} \\ &= \frac{k\lambda\alpha}{2} w \lim_{L \rightarrow \infty} \left\{ 6 - 6 + 4 \ln \left(\frac{1 + \sqrt{1 + \frac{y_1^2}{L^2}}}{y_1} \cdot \frac{y_2}{1 + \sqrt{1 + \frac{y_2^2}{L^2}}} \right) \right\} \\ &= 2k\lambda\alpha w \ln \left(\frac{y_2}{y_1} \right) \end{aligned}$$

This can be simplified by noting that current is the product of the drift speed and the charge density:

$$\begin{aligned} I &= \lambda v \\ \frac{dI}{dt} &= \lambda a = \lambda \alpha c^2 \\ \Rightarrow 2k\lambda\alpha &= \frac{2}{4\pi\epsilon_0 c^2} \frac{dI}{dt} \\ &= \frac{\epsilon_0 \mu_0}{2\pi} \frac{dI}{dt} \\ &= \frac{\mu_0}{2\pi} \frac{dI}{dt} \end{aligned}$$

This gives us:

$$\Delta V = \frac{\mu_0}{2\pi} \frac{dI}{dt} w \ln\left(\frac{y_2}{y_1}\right)$$

We'll return to this result again in Lesson 11.