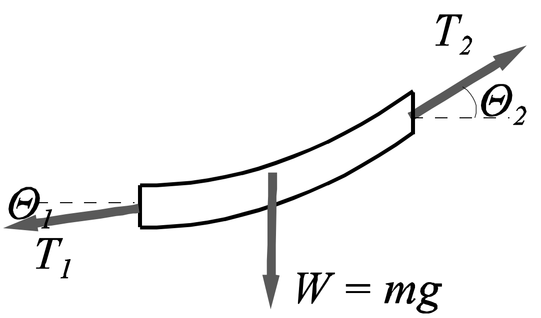
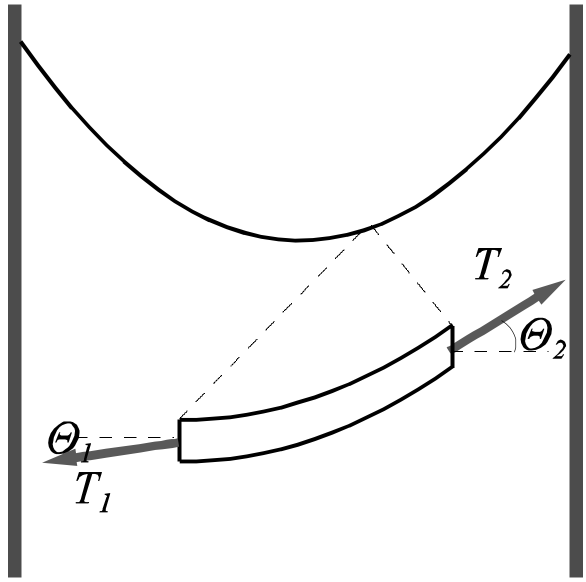
Physics 123 – Catenary problem

A chain is hung from two poles as shown.
Let T_1 = T, T_2 = T+ΔT , etc.





The equations of motion are:

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 - W = 0$$
$$T_2 \cos\theta_2 - T_1 \cos\theta_1 = 0$$
$$\text{Let} \quad T_1 = T, \quad T_2 = T + \Delta T$$
$$\theta_1 = \theta, \quad \theta_2 = \theta + \Delta\theta$$

Using the second equation,

$$T_2 \cos\theta_2 - T_1 \cos\theta_1 = 0$$
$$(T + \Delta T) \cos(\theta + \Delta\theta) - T\cos\theta = 0$$
$$(T + \Delta T) (\cos\theta \cos(\Delta\theta) - \sin\theta \sin(\Delta\theta)) - T\cos\theta = 0$$
$$(T + \Delta T) (\cos\theta - \sin\theta\ \Delta\theta) - T\cos\theta \approx 0$$
$$T\cos\theta + \cos\theta\ \Delta T - T\sin\theta\ \Delta\theta - T\cos\theta \approx 0$$
$$\cos\theta\ \Delta T - T\sin\theta\ \Delta\theta \approx 0$$
$$\frac{\Delta T}{\Delta\theta} \approx T \frac{\sin\theta}{\cos\theta}$$
$$\frac{dT}{d\theta} = T \tan\theta$$

We can use Maple to solve with the boundary condition that *$T(0)=T_0$*. The angle *$\theta=0$* corresponds to the place where the string is horizontal; that is, at the lowest point of the curve. Let us call this point *$(x,y)=(0,0)$*.

>e1:=diff(T(theta),theta)=T(theta)*tan(theta);

$$e1 := \frac{d}{d\theta} T(\theta) = T(\theta) \tan(\theta)$$

> b1:=T(0)=T0;

b1 := T(0) = T0

> dsolve([e1,b1],T(theta));

$$T(\theta) = \frac{T0}{\cos\theta}$$

So this equation gives us the tension as a function of the angle θ.

Now we look at the y equation:

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = W$$
$$(T + \Delta T)\sin(\theta + \Delta\theta) - T\sin\theta = \Delta m\, g$$
$$(T + \Delta T)(\sin\theta\cos(\Delta\theta) + \cos\theta\sin(\Delta\theta)) - T\sin\theta = \mu\,\Delta\ell\, g$$
$$(T + \Delta T)(\sin\theta + \cos\theta\,\Delta\theta) - T\sin\theta \approx \mu g\,\Delta\ell$$
$$T\sin\theta + \sin\theta\,\Delta T + T\cos\theta\,\Delta\theta - T\sin\theta \approx \mu g\,\Delta\ell$$
$$\sin\theta\,\Delta T + T\cos\theta\,\Delta\theta \approx \mu g\,\Delta\ell$$
$$\sin\theta\,\Delta T + \frac{T\cos\theta}{T\tan\theta}\Delta T \approx \mu g\,\Delta\ell \qquad \text{from the } x \text{ equation}$$
$$\sin\theta\,\Delta T + \frac{\cos^2\theta}{\sin\theta}\Delta T \approx \mu g\,\Delta\ell$$
$$(\sin^2\theta + \cos^2\theta)\,\Delta T \approx \mu g\,\Delta\ell\,\sin\theta$$
$$\Delta T \approx \mu g\,\Delta y$$
$$\frac{dT}{dy} = \mu g$$

This can be solved to give:

$$\frac{dT}{dy} = \mu g$$
$$T = \mu g y + T_0$$

In the end, though, we want y as a function of x. We can manipulate these equations in combination with the result that the slope is just $\tan\theta = \frac{dy}{dx}$. Let us begin by equating the two expressions for T:

$$T = \mu g y + T_0 = \frac{T_0}{\cos\theta}$$
$$y = \frac{T_0}{\mu g}\left(\frac{1}{\cos\theta} - 1\right)$$
$$\cos\theta = \frac{T_0}{T_0 + \mu g y}$$

Now we can express tan θ in terms of cos θ:

$$\frac{dy}{dx} = \tan\theta = \frac{\sqrt{1 - \cos^2\theta}}{\cos\theta}$$

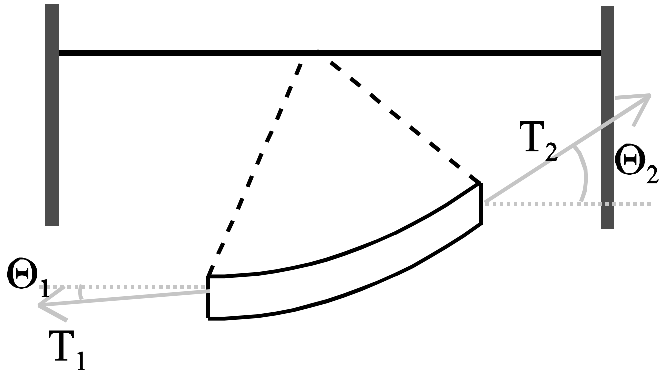
$$\frac{dy}{dx} = \sqrt{\frac{1}{\cos^2\theta} - 1}$$

$$\frac{dy}{dx} = \sqrt{\frac{(T_0 + \mu g y)^2}{T_0^{\ 2}} - 1}$$

We now put this into MAPLE with the boundary condition that *y(0)=0*.

See the Maple Worksheet http:webs.byu.edu/rees/123/main/maple.pdf for details.

Physics 123 – The Wave Equation



Assume a string of linear mass density μ is stretched between two poles. A wave is then produced on the string.

1. We assume that the weight is negligible.

2. We assume that the amplitude of the wave is small, θ is small. This means

$$\sin(\theta) \approx \theta$$
$$\cos(\theta) \approx 1$$
$$\sin(\theta + \Delta\theta) \approx \theta + \Delta\theta$$
$$\cos(\theta + \Delta\theta) \approx 1$$

4. We assume that the string can not move horizontally, so

$$(T + \Delta T)\cos(\theta + \Delta\theta) - T\cos\theta = 0$$
$$(T + \Delta T) - T\cos\theta = 0$$
$$\Delta T = 0$$

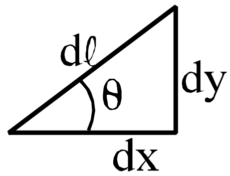
Thus, the tension in the string is essentially constant.

5. Since $d\ell \approx dx$, the mass of the string segment is approximately $dm = \mu\, dx$.

6. The equation of motion for the vertical direction gives:

$$(T + dT)\sin(\theta + d\theta) - T\sin\theta = dm\, a_y$$
$$T(\theta + d\theta) - T\theta \approx dm\, a_y$$
$$T d\theta = \mu dx \frac{\partial^2 y}{\partial t^2}$$
$$T \frac{d\theta}{dx} = \mu \frac{\partial^2 y}{\partial t^2}$$

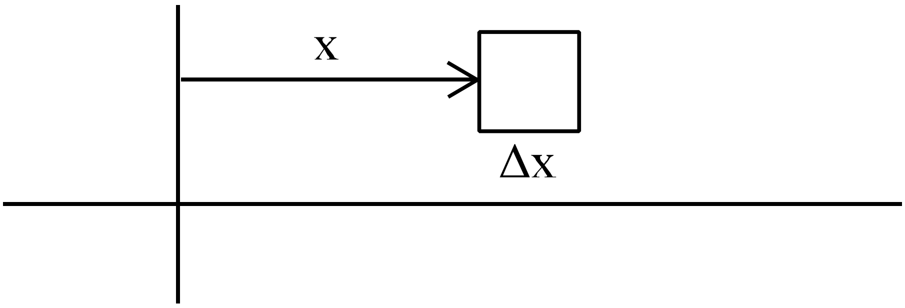
7. We now express θ in terms of the other variables: $\theta \approx \tan\theta = \frac{\partial y}{\partial x}$



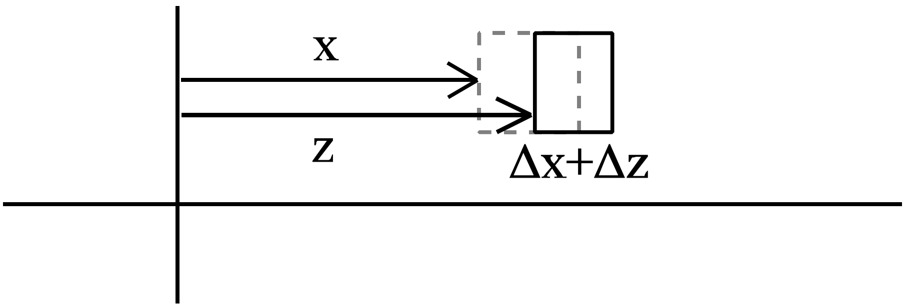
8. The expression in equation 6 becomes:

$$T \frac{d\theta}{dx} = \mu \frac{\partial^2 y}{\partial t^2}$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}$$

Physics 123 — The Wave Equations for Sound



Let us take a small box of air located, when no sound wave is present (at equilibrium), at a position *x*. The thickness of the box is Δ*x*. When a sound pulse travels through, the box is displaced and the pressure varies. If the pressure increases, the volume of the box will decrease. We will call the displacement from equilibrium *z*, and the change in thickness of the box Δ*z*. Thus:



(Note that we expect to show that $z(x,t) = B\sin(kx \pm \omega t + \varphi)$. Although this is what we expect to demonstrate, it may help you visualize the difference in the variables *x* and *z*.)

Let us call the cross-sectional area of the box *A*. The volume of the box is then just *A* Δ*x* initially. We can then write the change in volume as:

$$\Delta V = V_f - V_i = A(\Delta x + \Delta z) - A\,\Delta x = A\,\Delta z.$$

In the end, we want to change the variable Δ*z* to Δ*x*. To do this we make use of the "gradient."

$$\Delta z = \frac{\partial z}{\partial x}\Delta x$$

$$\Delta V = A\,\Delta z = A\frac{\partial z}{\partial x}\Delta x = V\frac{\partial z}{\partial x}$$

We now wish to find the relationship between pressure and volume. To do this we need two new definitions:

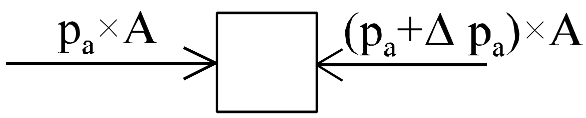
1) Acoustic pressure, p_a, is the difference between the pressure and atmospheric pressure. Thus $p_a = \Delta p = p - p_{atm}$.

2) Bulk modulus, K, is the ration of the change in pressure to the fractional change in the volume of a gas: $K = -\frac{\Delta p}{\Delta V / V}$. The minus sign indicates that the volume decreases as the pressure increases, so that K is positive.

Using these definitions, we have:

$$\Delta p = p_a = -K \frac{\Delta V}{V} = -K \frac{\partial z}{\partial x}.$$

Now we need to apply Newton's Laws to the block of air. We recall that the force on either side of the block is the pressure multiplied by the area. Thus the free body diagram is shown to the right. This gives:



$$F_{net} = -A \Delta p_a = ma.$$

We may use the identity $\Delta p_a = \frac{\partial p_a}{\partial x} \Delta x$ along with the definition of acceleration and the definition of mass density $\rho = m / V$ to obtain:

$$-A \frac{\partial p_a}{\partial x} \Delta x = \rho A \Delta x \frac{\partial^2 z}{\partial t^2}$$

$$-\frac{\partial p_a}{\partial x} = \rho \frac{\partial^2 z}{\partial t^2}$$

$$K \frac{\partial^2 z}{\partial x^2} = \rho \frac{\partial^2 z}{\partial t^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\rho}{K} \frac{\partial^2 z}{\partial t^2}$$

Thus we have a wave equation for the displacement of the box of air. The velocity of the wave is just

$$v = \sqrt{\frac{\rho}{K}}.$$

We could alternatively have found an equation in terms of pressure:

$$\frac{\partial p_a}{\partial x} = -\rho \frac{\partial^2 z}{\partial t^2}$$

$$\frac{\partial^2 p_a}{\partial x^2} = -\rho \frac{\partial^2}{\partial t^2} \frac{\partial z}{\partial x}$$

$$\frac{\partial^2 p_a}{\partial x^2} = -\rho \frac{\partial^2}{\partial t^2} \frac{-p_a}{K}$$

$$\frac{\partial^2 p_a}{\partial x^2} = \frac{\rho}{K} \frac{\partial^2 p_a}{\partial t^2}$$

Thus pressure satisfies a similar wave equation to displacement. Sound waves can therefore be considered as either pressure or displacement waves.

Finally, we can consider the wave passing through the air to be an adiabatic process. This leads to a useful expression for the velocity of sound in a gas.

$$pV^\gamma = C$$

$$p = cV^{-\gamma}$$

$$\frac{dp}{dV} = -\gamma C V^{-\gamma-1}$$

$$dp = p_a = -\gamma p \frac{dV}{V} = \gamma \frac{p_a}{K}$$

$$K = \gamma p$$

$$v = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{\gamma p}{\rho}} \approx \sqrt{\frac{\gamma p_0}{\rho}}$$

Algebraic Addition of Sine Functions

We wish to add two sine functions of the form

$$y_1 = B\sin(a+b), \qquad \text{and} \quad y_2 = C\sin(a+c).$$

We can do this either algebraically or geometrically. We will use both methods extensively, so you should become familiar with them.

First we do the algebraic method.

$$y_1 = B\sin(a+b), \qquad y_2 = C\sin(a+c)$$
$$y_1 = B(\sin a\cos b + \sin b\cos a), \qquad y_2 = C(\sin a\cos c + \sin c\cos a))$$
$$y_1 + y_2 = \sin a(B\cos b + C\cos c) + \cos a(B\sin b + C\sin c) \qquad \text{Eq(1)}$$

We can simplify this with a little trick.

$$\text{Let} \quad D^2 = (B\cos b + C\cos c)^2 + (B\sin b + C\sin c)^2$$
$$= B^2(\cos^2 b + \sin^2 b) + C^2(\cos^2 c + \sin^2 c) + 2BC(\cos b\cos c + \sin b\sin c)$$
$$= B^2 + C^2 + 2BC\cos(b-c)$$

If this is true, then we note that

$$\frac{(B\cos b + C\cos c)^2}{D^2} + \frac{(B\sin b + C\sin c)^2}{D^2} = \frac{D^2}{D^2} = 1$$

Because of this relationship, we can make the following trigonometric substitution:

$$\cos d \equiv \frac{B\cos b + C\cos c}{D}, \qquad \sin d \equiv \frac{B\sin b + C\sin c}{D}.$$

We can now put this in Eq(1) to simplify it.

$$y_1 + y_2 = \sin a(B\cos b + C\cos c) + \cos a(B\sin b + C\sin c)$$
$$= D\sin a\cos d + D\cos a\sin d \qquad \text{Eq(2)}$$
$$= D\sin(a+d)$$

where, as we have shown above:

$$D^2 = B^2 + C^2 + 2BC\cos(b-c), \qquad \tan d \equiv \frac{B\sin b + C\sin c}{B\cos b + C\cos c}$$

Geometric Addition of Sine Functions

We showed that if we add the two waves

$$y_1 = B\sin(a+b), \qquad \text{and} \quad y_2 = C\sin(a+c),$$

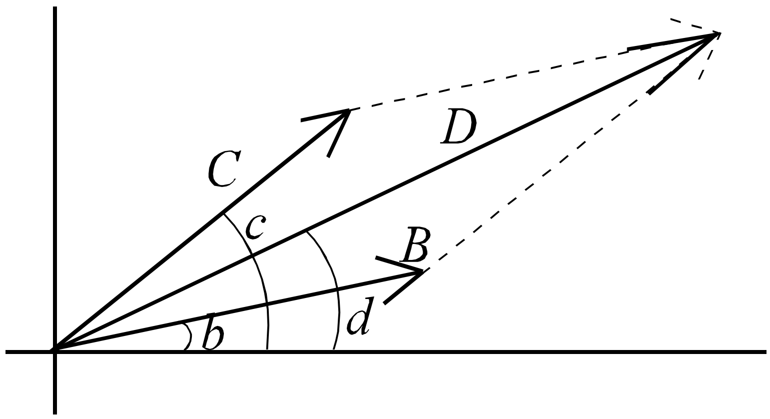
the result is (Eq(1) and Eq(2)):

$$\begin{aligned} y_1 + y_2 &= \sin a\,(B\cos b + C\cos c) + \cos a\,(B\sin b + C\sin c) \\ &= D\sin a\cos d + D\cos a\sin d \end{aligned}$$

We see that these expressions are equal if

$$\begin{aligned} D\cos d &= B\cos b + C\cos c \\ D\sin d &= B\sin b + C\sin c \end{aligned}$$

Note that this looks like adding components of vectors! So let's make these into vectors:



Note then that

$$D_x = B_x + C_x, \quad D_y = B_y + C_y, \quad D^2 = D_x^2 + D_y^2, \quad \tan d = \frac{D_y}{D_x}$$

The vectors are called "phasors." Note the following characteristics of phasors:

- A phasor is not a "drawing" of the wave. The direction is not the direction the wave travels. It is an abstract way to represent the amplitude and phase of a wave.
- The magnitude of a phasor is the **amplitude** of the wave.
- The angle of a phasor is determined by the **phase** of the wave.
- The part of the phase a is common to all the waves. It does not affect the phasor.
- The angle of the phasor is whatever is left over from the overall phase after subtracting off the common term a.

Some examples of how to choose phase angles may be helpful:

1) To add $y_1 = A\sin(kx - \omega t)$ and $y_2 = B\sin(kx - \omega t + \varphi)$, choose $a = kx - \omega t$, so the phasors have angles 0 and φ.

2) To add $y_1 = A\sin(kx - \omega t)$ and $y_2 = B\sin(kx + \omega t)$, choose $a = kx$, so the phasors have angles $-\omega t$ and $+\omega t$. (Note as t increases, the phasor of waves traveling to the right rotate clockwise and waves traveling left rotate counterclockwise.)

3) To add $y_1 = A\sin(kr_1 - \omega t + \varphi)$ and $y_2 = B\sin(kr_2 - \omega t + \varphi)$ where the waves are in three dimensions and the positions r_1 and r_2 are measured from different sources to a fixed point P in space, choose $a = -\omega t + \varphi$, so the phasors have angles kr_1 and kr_2.

4) One can always choose $a = 0$, so the phase angles are the overall phases of the waves.