

Homework 5-1

1. Write the following equations in matrix form:

$$(a) \quad \begin{cases} 3x+5z=7 \\ 4y+2z=5 \\ 3x+2y+2z=0 \end{cases} \quad \begin{pmatrix} 3 & 0 & 5 \\ 0 & 4 & 2 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix}$$

$$(b) \quad \begin{cases} 1x+2y+3z=1 \\ 4x+5y+6z=2 \\ 7x+8y+9z=3 \end{cases} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2. The “transpose” operation changes a column vector to a row vector and visa-versa.

$$(a) \text{ Find } [(1 \ 2 \ 3 \ 4 \ 5)^T]^T \quad [(1 \ 2 \ 3 \ 4 \ 5)^T]^T = (1 \ 2 \ 3 \ 4 \ 5)$$

$$(b) \text{ Find } \begin{pmatrix} 3 & 0 & 5 \\ 0 & 4 & 2 \\ 3 & 2 & 2 \end{pmatrix}^T \quad \begin{pmatrix} 3 & 0 & 5 \\ 0 & 4 & 7 \\ 3 & 2 & 6 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 4 & 2 \\ 5 & 7 & 6 \end{pmatrix}$$

2. Find the dot product of the following vectors:

$$A = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \quad A \cdot B = A^T B = (1 \ 5 \ 7) \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 30$$

3. Calculate the following matrix products:

$$(a) \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 5 \\ 3 & 2 & 1 \\ 2 & 4 & 6 \end{pmatrix}$$

$$(b) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ -4 & -5 & -6 \end{pmatrix}$$

4. Construct a matrix which does a mirror reflection in the z direction.

$$M_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

5. Show that the matrix you found in Problem 4 is its own inverse.

$$M_z M_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. Find a matrix which rotates a three-dimensional vector by $+90^\circ$ around the z axis.

We know that such a rotation will leave the z coordinate of a vector unchanged, it will make the original x component the new y component, and it will make $-1 \times$ the original y component the new x component. Thus:

$$\mathbf{R}_{+z} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R}_{+z} \mathbf{r} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ -x \\ z \end{pmatrix}$$

7. A general rotation about the z axis by an angle θ can be shown to be:

$$\mathbf{R}_{+z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show what this matrix operation does to an arbitrary vector \mathbf{r} . Does this result seem correct?

$$\mathbf{R}_{+z}(\theta) \mathbf{r} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \\ z \end{pmatrix}.$$

8. Show that the inverse of the matrix given in Problem 7 is just a rotation about the z axis of $-\theta$.

Note that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Then:

$$\begin{aligned} \mathbf{R}_{+z}(\theta) \mathbf{R}_{+z}(-\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta \sin\theta - \cos\theta \sin\theta & 0 \\ \cos\theta \sin\theta - \cos\theta \sin\theta & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

9. Find the matrix which gives a boost in the y direction.

$$\mathbf{B}_{+x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

10. (Optional) Find a boost in a direction of 45° from the x axis toward the y axis. Use rotation matrices.

We need to rotate about the z axis by 45° . The matrix for this is given in Problem 7 to be:

$$\mathbf{R}_{+z}(45^\circ) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

However, we need to recast this into four-dimensional space. Since the rotation does nothing to the time component, we have:

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

According to Section 1-6, we have:

$$\begin{aligned} \tilde{\mathbf{B}} = \mathbf{R} \mathbf{B}_x \mathbf{R}^{-1} &= \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ -\sqrt{2}v/c & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ -\sqrt{2}v/c & 2 & 0 & 0 \\ -\sqrt{2}v/c & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{v}{\sqrt{2}c} & 1 & 0 & 0 \\ -\frac{v}{\sqrt{2}c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Homework 5-2

1. On the grid by doing the following: provided, draw the trajectory for a particle initially traveling horizontally in a gravitational field. Use the non-relativistic energy-momentum vector as the propagator. Plot only the x and y directions on the graph.

Follow these instructions:

a. The particle starts at the position indicated.

b. The x component of the propagator is 5 units in length. There is no x component of force, so the x component of the propagator remains constant.

c. The y component of the propagator is initially zero. The y component of the force is constant, so the *change* in the y component of the propagator remains constant. (We take this to be one unit downward with each step. See (e) below.)

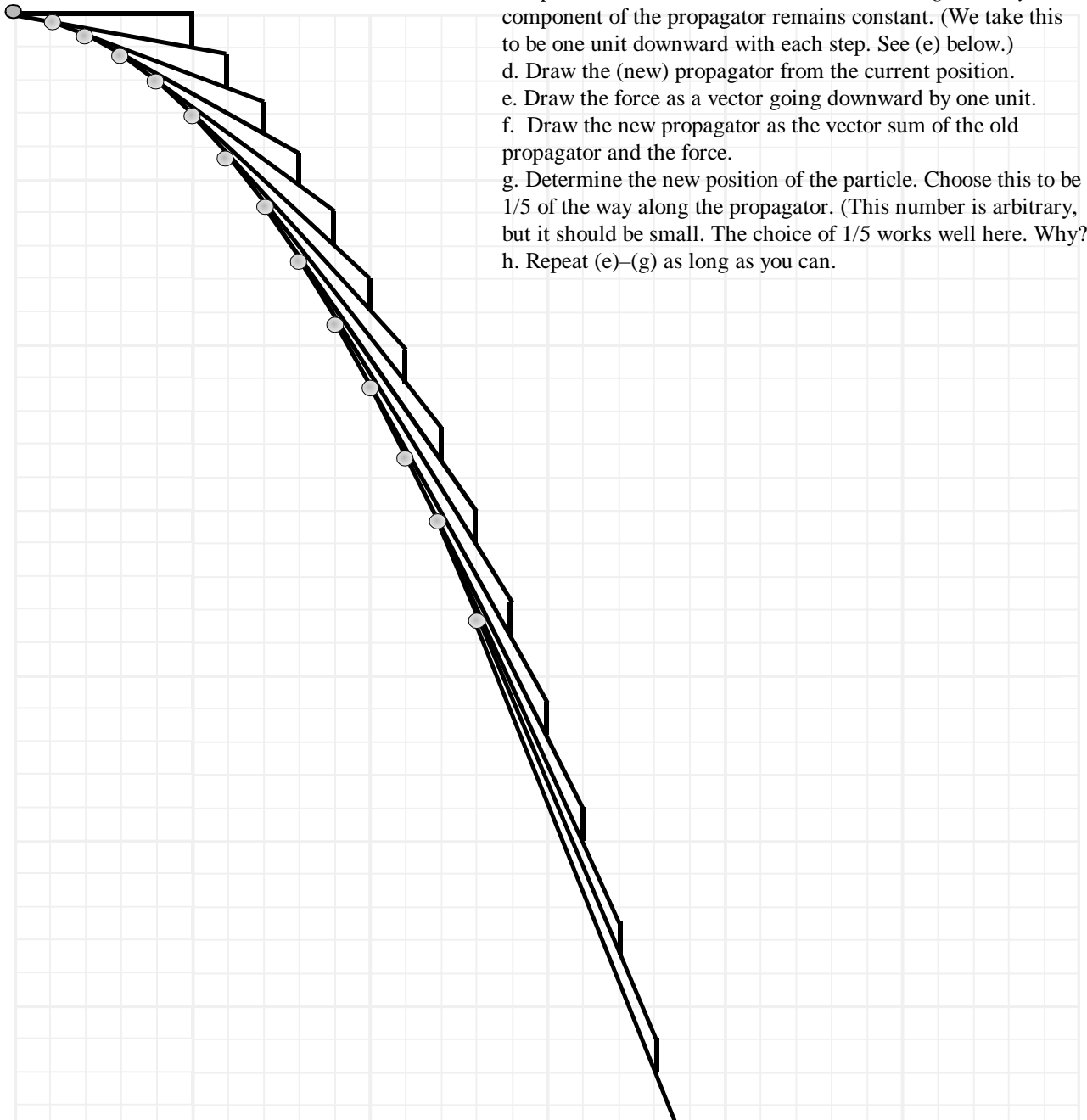
d. Draw the (new) propagator from the current position.

e. Draw the force as a vector going downward by one unit.

f. Draw the new propagator as the vector sum of the old propagator and the force.

g. Determine the new position of the particle. Choose this to be $1/5$ of the way along the propagator. (This number is arbitrary, but it should be small. The choice of $1/5$ works well here. Why?)

h. Repeat (e)–(g) as long as you can.



2. Now, let's think about what Problem 1 means. Keep in mind the following relationships for non-relativistic propagation:

$$x' = x + \varepsilon \mathbf{E} \quad \mathbf{x} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} mc^2 \\ mv_x c \\ mv_y c \\ mv_z c \end{pmatrix} \equiv \begin{pmatrix} E_0 \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \quad \Delta \mathbf{E} = \begin{pmatrix} 0 \\ F_x \\ F_y \\ F_z \end{pmatrix} \varepsilon E_0$$

Let unit on the graph of Problem 1 to be 1 meter for distances and 1 J for pc . Also let $c \equiv 1$ m/s.

Find the following quantities (be sure to include units):

a. ε ε is the fraction of \mathbf{E} we add to \mathbf{x} to get \mathbf{x}' . Hence $\varepsilon = 1 / 5$ m/J.
Note that \mathbf{E} has units of J and \mathbf{x} has units of m.

b. p_x $p_x c = 5$ J, so $p_x = 5$ kg m/s.

c. E_0 Hint: use $\Delta p_y c = 1$ J. (Where does this come from?) Also remember $E_0 = mc^2$.

$$1 \text{ J} = \Delta p_y c = F_y \varepsilon E_0 = mg \varepsilon E_0 = \frac{E_0}{c^2} \varepsilon g E_0$$

$$E_0^2 = 1 \text{ J} \frac{c^2}{\varepsilon g}$$

$$E_0 = 0.714 \text{ J}$$

d. Δct , the step size in the time dimension.

$$\text{Note that } \Delta \mathbf{r} = \varepsilon \mathbf{p}, \text{ so } \Delta ct = \varepsilon E_0 = 0.143 \text{ m.}$$

e. What is the mass of the particle?

$$E_0 = mc^2, \text{ so } m = 0.714 \text{ kg.}$$

f. What is the initial velocity of the particle?

$$p_x c = mv_x c = 5 \text{ J}, \quad v_x = \frac{p_x}{m} = 7.00 \text{ m/s.}$$

g. Estimate the final velocity of the particle (the last one you are able to determine.)

The final value of pc is about 16 squares = 16 J. Therefore

$$p_f c = mv_f c = 16 \text{ J}, \quad v_f = \frac{p_f}{m} = 22.5 \text{ m/s.}$$

3. If the time dimension were to go back into the page, describe the appearance of the world line of the particle of Problem 1 (still taking the problem to be non-relativistic).

Each time step is $\Delta w = \epsilon E_0$, which remains constant. This means that each successive point would be an equal distance behind the previous point in the time dimension. If the scale for w is the same as for x and y , there would be about seven data points per unit displacement in w .

4. If the motion of the particle in Problem 1 were relativistic. For sake of precision, let the initial values be: $\gamma = 3$, the x component of the propagator 5 units to the right, and the change in the y component of the propagator 1 unit down. How would the propagator differ from the non-relativistic case? How would the world line differ?

Each component of the propagator would be γ times the corresponding non-relativistic result. To begin with, the propagator would be just the same, though. As the particle accelerates and the velocity increases, the mass would increase, causing the time step and spatial steps to constantly grow longer compared to the non-relativistic case.

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Homework 5-3

1. The rest energy of an electron is 0.511 MeV. What is the momentum of an electron traveling at 1% of the speed of light? At 99% of the speed of light? How does this compare to the non-relativistic results? You can express your answer in units of MeV/c.

Non-relativistically $p=mv=m\beta c=E_0\beta/c$ Hence the momenta are 5.11 keV/c and 0.506 MeV/c.

Relativistically $p=m\gamma v=m\gamma\beta c=E_0\gamma\beta/c$ Hence the momenta are 5.11 keV/c and 3.59 MeV/c.

2. What is the kinetic energy and the total energy of the electrons described in Problem 1? Compute the kinetic energies both relativistically and non-relativistically.

Relativistically $E=E_0\gamma$. This is 0.511 MeV and 3.62 MeV.

$$K=E-E_0=25.6 \text{ eV and } 3.11 \text{ MeV.}$$

Non-relativistically $K = \frac{1}{2}mv^2 = \frac{1}{2}m\beta^2 c^2 = \frac{1}{2}E_0\beta^2 = 25.6 \text{ MeV/c and } 0.250 \text{ MeV/c.}$

3. The maximum speed of a moving particle is c . Is there a maximum kinetic energy? Briefly explain.

No, there is no maximum kinetic energy because as v approaches c , γ becomes infinite. Note $K = E_0 (\gamma - 1)$.

4. (a) If the momentum of a particle doubles, by how much does its speed change?

$$\begin{aligned} p_2 c &= 2p_1 c, & E_0 \beta_2 \gamma_2 &= 2E_0 \beta_1 \gamma_1 \\ \beta_2^2 \gamma_2^2 &= 4\beta_1^2 \gamma_1^2, & \frac{\beta_2^2}{1-\beta_2^2} &= 4\beta_1^2 \gamma_1^2 \\ \beta_2^2 &= 4\beta_1^2 \gamma_1^2 - 4\beta_2^2 \beta_1^2 \gamma_1^2, & \beta_2^2 (1 + 4\beta_1^2 \gamma_1^2) &= 4\beta_1^2 \gamma_1^2 \\ \beta_2^2 &= \frac{4\beta_1^2 \gamma_1^2}{1 + 4\beta_1^2 \gamma_1^2}, & \beta_2^2 &= \frac{4\beta_1^2}{\frac{1}{\gamma_1^2} + 4\beta_1^2} \\ \beta_2^2 &= \frac{4\beta_1^2}{1 - \beta_1^2 + 4\beta_1^2}, & \beta_2^2 &= \frac{4\beta_1^2}{1 + 3\beta_1^2} \\ \beta_2 &= 2 \frac{\beta_1}{\sqrt{1 + 3\beta_1^2}} \end{aligned}$$

Note that when the speed is small, the speed essentially double. However, as the speed approaches c , the velocity remains the same.

(b) If the speed of a particle doubles, by how much how does its momentum change?

$$\begin{aligned} p_2 c &= E_0 \beta_2 \gamma_2 = 2E_0 \beta_1 \frac{1}{\sqrt{1 - 4\beta_1^2}} \\ p_2 c &= 2E_0 \beta_1 \gamma_1 \sqrt{\frac{1 - \beta^2}{1 - 4\beta_1^2}} \\ p_2 c &= 2p_1 c \sqrt{\frac{1 - \beta^2}{1 - 4\beta_1^2}} \end{aligned}$$

Note that the denominator blows up if the speed of the particle is half the speed of light. That is, the velocity can double only if the particle is traveling slower than half the speed of light. Near this value, the momentum gets very large.

(c) If the speed of a particle doubles, by how much does its total energy change?

$$E_2 = E_0 \gamma_2 = E_0 \frac{1}{\sqrt{1-4\beta_1^2}}$$

$$E_2 = E_0 \gamma_1 \sqrt{\frac{1-\beta^2}{1-4\beta_1^2}}$$

$$E_2 = E_1 \sqrt{\frac{1-\beta^2}{1-4\beta_1^2}}$$

When the velocity is small, the total energy only changes a small amount because it is dominated by the rest energy.

(d) If the speed of a particle doubles, by how much does its kinetic energy change? (Note that non-relativistically, its kinetic energy becomes four times larger.)

$$K_2 = E_0(\gamma_2 - 1) = E_0(\gamma_1 - 1) \frac{(\gamma_2 - 1)}{(\gamma_1 - 1)}$$

$$K_2 = K_1 \frac{\sqrt{\frac{1}{1-4\beta_1^2}} - 1}{\sqrt{\frac{1}{1-\beta_1^2}} - 1}$$

When the velocity is small, we may use a Taylor series to approximate the square roots:

$$K_2 = K_1 \frac{(1-4\beta_1^2)^{-1/2} - 1}{(1-\beta_1^2)^{-1/2} - 1}$$

$$K_2 \approx K_1 \frac{1+2\beta_1^2 - 1}{1+\frac{1}{2}\beta_1^2 - 1}$$

$$K_2 \approx 4K_1$$

5. A 250.0 MeV proton (e.g., a proton with a kinetic energy of 250.0 MeV) strikes a carbon nucleus at rest. The proton is detected with a kinetic energy of 235.0 MeV. At what angles with respect to the incident proton's direction do the two particles emerge? The rest energy of a proton is 938.28 MeV and the rest energy of the carbon nucleus is 11174.8 MeV. Hint: Use conservation of each component of the energy-momentum four vector. Assume that the incident proton moves in the z direction and after the collision, the protons emerge in the x - z plane.

$$\begin{pmatrix} E_p \\ 0 \\ 0 \\ p_p c \end{pmatrix} + \begin{pmatrix} E_{0c} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E'_p \\ p'_p c \sin \theta_p \\ 0 \\ p'_p c \cos \theta_p \end{pmatrix} + \begin{pmatrix} E'_c \\ p'_c c \sin \theta_c \\ 0 \\ p'_c c \cos \theta_c \end{pmatrix}$$

Starting with the energy equation:

$$\begin{aligned} E_p + E_{0c} &= E'_p + E'_c \\ K_p &= K'_p + K'_c \\ K'_c &= K_p - K'_p = 15.0 \text{ MeV} \end{aligned}$$

This allows us to find the total energies and momenta:

$$\begin{aligned} E &= K + E_0, & E^2 &= p^2 c^2 + E_0^2 \\ E_p &= 1188.3 \text{ MeV}, & p_p &= 729.1 \text{ MeV}/c \\ E'_p &= 1173.3 \text{ MeV}, & p'_p &= 704.4 \text{ MeV}/c \\ E'_c &= 11189.8 \text{ MeV}, & p_c &= 579.2 \text{ MeV}/c \end{aligned}$$

Now we may apply the momentum equations:

$$\begin{aligned} 0 &= p'_p c \sin \theta_p + p'_c c \sin \theta_c & p_p c &= p'_p c \cos \theta_p + p'_c c \cos \theta_c \\ p'_c c \sin \theta_c &= -p'_p c \sin \theta_p & p'_c c \cos \theta_c &= p_p c - p'_p c \cos \theta_p \\ p'^2_c c^2 &= p'^2_c c^2 \sin^2 \theta_c + p'^2_c c^2 \cos^2 \theta_c & &= p'^2_p c^2 \sin^2 \theta_p + (p_p c - p'_p c \cos \theta_p)^2 \\ p'^2_c c^2 &= p'^2_p c^2 \sin^2 \theta_p + p_p^2 c^2 - 2p_p p'_p c^2 \cos \theta_p + p'^2_p c^2 \cos^2 \theta_p \\ p'^2_c c^2 &= p'^2_p c^2 + p_p^2 c^2 - 2p_p p'_p c^2 \cos \theta_p \\ \cos \theta_p &= \frac{p'^2_p + p_p^2 - p'^2_c}{2p_p p'_p} \end{aligned}$$

This leads to $\theta_p = 47.6^\circ$, and $\theta_c = -64.0^\circ$.

Homework 5-4

1. Write down the boost that takes four vectors from a frame \mathbf{S} to a frame \mathbf{S}' moving at velocity $v = \beta c \hat{y}$ with respect to \mathbf{S} .

$$\mathbf{L}(\beta c \hat{y}) = \begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Write down the inverse of the matrix you found for Problem 1. Show explicitly that this matrix is the inverse.

$$\mathbf{L}^{-1}(\beta c \hat{y}) = \mathbf{L}(-\beta c \hat{y}) = \begin{pmatrix} \gamma & 0 & \beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & \beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma^2(1-\beta^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. (This problem has a lot of algebra. It is important to know how to do the algebra, but if you understand the algebra, I don't mind if you trust parts of my solution rather than working through the details. Be sure you understand the ideas, though.)

A particle travels in frame \mathbf{S} with identical velocities in the x and y directions and with zero velocity in the z direction.

$$\mathbf{E} = \begin{pmatrix} E_0\gamma \\ \frac{1}{\sqrt{2}} E_0\beta\gamma \\ \frac{1}{\sqrt{2}} E_0\beta\gamma \\ 0 \end{pmatrix} = \frac{E_0\gamma}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \\ \beta \\ \beta \\ 0 \end{pmatrix}$$

Find the following matrices:

(a) Find a boost, \mathbf{L} , that in one step brings the particle to its rest frame, \mathbf{S}'' . (Use Eq. 4-3. You can just write it down by inspection.)

$$\mathbf{L} = \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}\beta\gamma & -\frac{1}{\sqrt{2}}\beta\gamma & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma & \frac{\gamma+1}{2} & \frac{\gamma-1}{2} & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma & \frac{\gamma-1}{2} & \frac{\gamma+1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Find a boost, \mathbf{L}_y , in the y direction that brings the y component of the velocity to zero in the \mathbf{S}' frame. Express the matrix in terms of β_y and γ_y , then evaluate β_y and γ_y .

$$\mathbf{L}_y = \begin{pmatrix} \gamma_y & 0 & -\beta_y \gamma_y & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_y \gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To find the boost parameters:

$$\begin{pmatrix} \gamma_y & 0 & -\beta_y \gamma_y & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_y \gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{E_0 \gamma}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \\ \beta \\ \beta \\ 0 \end{pmatrix} = \frac{E_0 \gamma}{\sqrt{2}} \begin{pmatrix} \gamma_y \sqrt{2} - \beta_y \gamma_y \beta \\ \beta \\ -\beta_y \gamma_y \sqrt{2} + \gamma_y \beta \\ 0 \end{pmatrix} = \begin{pmatrix} E_0 \gamma_x \\ E_0 \beta_x \gamma_x \\ 0 \\ 0 \end{pmatrix}$$

We write the last vector because we know that the resultant vector must have this general form with the y component of velocity zero. We'll find β_y and γ_y later.

The third line gives: $\beta_y = \beta/\sqrt{2}$, and in turn,

$$\gamma_y = \frac{1}{\sqrt{1-\beta_y^2}} = \frac{1}{\sqrt{1-\frac{1}{2}\beta^2}}$$

(c) Find a boost, \mathbf{L}_x , in the x direction that brings the particle to its rest frame, \mathbf{S}'' . Express the matrix in terms of β_x and γ_x , then evaluate β_x and γ_x .

First, we turn to the matrix equation above. From the first line, we have

$$\gamma_x = \frac{\gamma}{\sqrt{2}} \gamma_y (\sqrt{2} - \beta \beta_y) = \frac{\gamma}{\sqrt{2}} \frac{1}{\sqrt{1-\frac{1}{2}\beta^2}} \sqrt{2} (1 - \frac{1}{2}\beta^2) = \gamma \sqrt{1-\frac{1}{2}\beta^2} = \frac{\gamma}{\gamma_y}$$

And from the second line,

$$\beta_x \gamma_x = \frac{1}{\sqrt{2}} \beta \gamma, \quad \beta_x = \frac{1}{\sqrt{2}} \beta \gamma_y$$

The matrix is a simple boost in the x direction:

$$\mathbf{L}_x = \begin{pmatrix} \gamma_x & -\beta_x \gamma_x & 0 & 0 \\ -\beta_x \gamma_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) Find the product of the boosts $\mathbf{L}_x \mathbf{L}_y$. Note that this is not the same as \mathbf{L} . It is convenient to express your answer in terms of β , γ , and γ_y .

$$\mathbf{L}_x \mathbf{L}_y = \begin{pmatrix} \gamma_x & -\beta_x \gamma_x & 0 & 0 \\ -\beta_x \gamma_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_y & 0 & -\beta_y \gamma_y & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_y \gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_x \gamma_y & -\beta_x \gamma_x & -\gamma_x \beta_y \gamma_y & 0 \\ -\gamma_y \beta_x \gamma_x & \gamma_x & \beta_x \beta_y \gamma_x \gamma_y & 0 \\ -\beta_y \gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Substituting the values we have above,

$$\mathbf{L}_x \mathbf{L}_y = \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}} \beta \gamma & -\frac{1}{\sqrt{2}} \beta \gamma & 0 \\ -\frac{1}{\sqrt{2}} \beta \gamma \gamma_y & \frac{\gamma}{\gamma_y} & \frac{1}{2} \beta^2 \gamma \gamma_y & 0 \\ -\frac{1}{\sqrt{2}} \beta \gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(e) (Optional) We can define a rotation matrix about the z axis with a rotation angle θ by

$$\mathbf{R}_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \cos\theta = \frac{\gamma_y(\gamma+1)}{2\gamma}, \quad \sin\theta = \frac{\gamma_y(\gamma-1)}{2\gamma}$$

Show that $\mathbf{L} = \mathbf{R}_z \mathbf{L}_x \mathbf{L}_y$. (That is, the successive boosts are equivalent to a pure boost followed by a rotation.)

Two identities may simplify your algebra. Proofs follow:

$$\frac{\gamma^2 - 1}{\gamma^2} = 1 - \frac{1}{\gamma^2} = 1 - (1 - \beta^2) = \beta^2 = \frac{(\gamma - 1)(\gamma + 1)}{\gamma^2} \quad \frac{\gamma^2 + 1}{\gamma^2} = 1 + \frac{1}{\gamma^2} = 1 + (1 - \beta^2) = 2 - \beta^2 = 2(1 - \frac{1}{2}\beta^2) = \frac{2}{\gamma_y^2}$$

$$\mathbf{R}_z \mathbf{L}_x \mathbf{L}_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma_y(\gamma+1)}{2\gamma} & -\frac{\gamma_y(\gamma-1)}{2\gamma} & 0 \\ 0 & \frac{\gamma_y(\gamma-1)}{2\gamma} & \frac{\gamma_y(\gamma+1)}{2\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}} \beta \gamma & -\frac{1}{\sqrt{2}} \beta \gamma & 0 \\ -\frac{1}{\sqrt{2}} \beta \gamma \gamma_y & \frac{\gamma}{\gamma_y} & \frac{1}{2} \beta^2 \gamma \gamma_y & 0 \\ -\frac{1}{\sqrt{2}} \beta \gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}\beta\gamma & -\frac{1}{\sqrt{2}}\beta\gamma & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma\gamma_y \frac{\gamma_y(\gamma+1)}{2\gamma} + \frac{1}{\sqrt{2}}\beta\gamma_y \frac{\gamma_y(\gamma-1)}{2\gamma} & \frac{\gamma}{\gamma_y} \frac{\gamma_y(\gamma+1)}{2\gamma} & \frac{1}{2}\beta^2\gamma\gamma_y \frac{\gamma_y(\gamma+1)}{2\gamma} - \gamma_y \frac{\gamma_y(\gamma-1)}{2\gamma} & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma\gamma_y \frac{\gamma_y(\gamma-1)}{2\gamma} - \frac{1}{\sqrt{2}}\beta\gamma_y \frac{\gamma_y(\gamma+1)}{2\gamma} & \frac{\gamma}{\gamma_y} \frac{\gamma_y(\gamma-1)}{2\gamma} & \frac{1}{2}\beta^2\gamma\gamma_y \frac{\gamma_y(\gamma-1)}{2\gamma} + \gamma_y \frac{\gamma_y(\gamma+1)}{2\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}\beta\gamma & -\frac{1}{\sqrt{2}}\beta\gamma & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma \frac{\gamma_y^2}{2\gamma^2} (\gamma^2+1) & \frac{1}{2}(\gamma+1) & \frac{1}{4}\gamma_y^2\beta^2(\gamma+1) - \gamma_y^2 \frac{(\gamma-1)}{2\gamma} & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma \frac{\gamma_y^2}{2\gamma^2} (\gamma^2+1) & \frac{1}{2}(\gamma-1) & \frac{1}{4}\gamma_y^2\beta^2(\gamma-1) + \gamma_y^2 \frac{(\gamma+1)}{2\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}\beta\gamma & -\frac{1}{\sqrt{2}}\beta\gamma & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma \frac{\gamma_y^2}{\gamma_y^2} & \frac{1}{2}(\gamma+1) & \gamma_y^2 \frac{(\gamma-1)(\gamma+1)^2}{4\gamma^2} - \gamma_y^2 \frac{(\gamma-1)}{2\gamma} & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma \frac{\gamma_y^2}{\gamma_y^2} & \frac{1}{2}(\gamma-1) & \gamma_y^2 \frac{(\gamma+1)(\gamma-1)^2}{4\gamma^2} + \gamma_y^2 \frac{(\gamma+1)}{2\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}\beta\gamma & -\frac{1}{\sqrt{2}}\beta\gamma & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma & \frac{1}{2}(\gamma+1) & \gamma_y^2 \frac{(\gamma-1)}{2\gamma} \left(\frac{(\gamma+1)^2}{2\gamma} - 1 \right) & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma & \frac{1}{2}(\gamma-1) & \gamma_y^2 \frac{(\gamma+1)}{2\gamma} \left(\frac{(\gamma-1)^2}{2\gamma} + 1 \right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}\beta\gamma & -\frac{1}{\sqrt{2}}\beta\gamma & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma & \frac{1}{2}(\gamma+1) & \gamma_y^2 \frac{(\gamma-1)}{2} \frac{\gamma^2+1}{2\gamma^2} & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma & \frac{1}{2}(\gamma-1) & \gamma_y^2 \frac{(\gamma+1)}{2} \frac{\gamma^2+1}{2\gamma^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\frac{1}{\sqrt{2}}\beta\gamma & -\frac{1}{\sqrt{2}}\beta\gamma & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma & \frac{\gamma+1}{2} & \frac{\gamma-1}{2} & 0 \\ -\frac{1}{\sqrt{2}}\beta\gamma & \frac{\gamma-1}{2} & \frac{\gamma+1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Homework 5-5

1. A proton strikes a carbon nucleus at rest. The initial kinetic energy of the proton is K . Assume the rest masses of the proton, E_{0p} , and carbon, E_{0c} , are known.

(a) Find expressions for the following quantities for the proton before the collision in terms of K and E_{0p} .

$$\begin{aligned}
 \text{Total energy, } E_p & E_p = K + E_{0p} \\
 \gamma_p & \gamma_p = \frac{E_p}{E_{0p}} = 1 + \frac{K}{E_{0p}} \\
 \beta_p & \beta_p = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{E_{0p}^2}{E_p^2}} = \sqrt{1 - \frac{E_{0p}^2}{(E_{0p} + K)^2}} = \frac{\sqrt{K^2 + 2KE_{0p}}}{E_{0p} + K} \\
 p_i & p_i c = \sqrt{E_p^2 - E_{0p}^2}, \quad p_i = \frac{1}{c} \sqrt{K^2 + 2KE_{0p}}
 \end{aligned}$$

(b) Find a vector that represents the energy-momentum of the system prior to the collision. In this and succeeding parts of the problem, you may leave your answers in terms of any quantity that has been found in terms of the givens. Assume that the proton moves in the z direction.

$$\mathbf{E} = \begin{pmatrix} E_i \\ 0 \\ 0 \\ p_i c \end{pmatrix} = \begin{pmatrix} E_p + E_{0c} \\ 0 \\ 0 \\ p_i c \end{pmatrix}$$

(c) Find the Lorentz transformations that takes the four-vector of part (b) to the inertial frame in which the total momentum is zero.

$$\mathbf{E}' = \mathbf{L} \mathbf{E}$$

$$\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E_i \\ 0 \\ 0 \\ p_i c \end{pmatrix} = \begin{pmatrix} \gamma E_i - \beta\gamma p_i c \\ 0 \\ 0 \\ \gamma p_i c - \beta\gamma E_i \end{pmatrix} = \begin{pmatrix} W \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

W is the total energy in the zero momentum (center of mass) frame. To find β and γ :

$$\begin{aligned}
 \gamma p_i c - \beta\gamma E_i &= 0 \\
 \beta &= \frac{p_i c}{E_i} \\
 \gamma^2 &= \frac{1}{1 - \frac{p_i^2 c^2}{E_i^2}} = \frac{E_i^2}{E_i^2 - p_i^2 c^2} \\
 \gamma &= \frac{E_i}{\sqrt{E_i^2 - p_i^2 c^2}}
 \end{aligned}$$

(d) Find the total energy of the system in the zero momentum frame, W , and show that the Lorentz transformation can be written

$$L = \begin{pmatrix} \frac{E_i}{W} & 0 & 0 & -\frac{p_i c}{W} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{p_i c}{W} & 0 & 0 & \frac{E_i}{W} \end{pmatrix} = \frac{W = \gamma E_i - \beta \gamma p_i c}{\sqrt{E_i^2 - p_i^2 c^2} - \frac{p_i^2 c^2}{\sqrt{E_i^2 - p_i^2 c^2}} \frac{E_i}{\sqrt{E_i^2 - p_i^2 c^2}}} \gamma = \frac{E_i}{W}, \quad \beta \gamma = \frac{p_i c}{W}$$

$$W = \sqrt{E_i^2 - p_i^2 c^2}$$

(e) Boost each of the initial energy-momentum four-vectors into the zero momentum frame in order to find the total energy and momentum the proton and the carbon in this frame.

$$\begin{pmatrix} E'_p \\ 0 \\ 0 \\ p'_p c \end{pmatrix} = \frac{1}{W} \begin{pmatrix} E_i & 0 & 0 & -p_i c \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 0 \\ -p_i c & 0 & 0 & E_i \end{pmatrix} \begin{pmatrix} E_p \\ 0 \\ 0 \\ p_i c \end{pmatrix} = \frac{1}{W} \begin{pmatrix} E_i E_p - p_i^2 c^2 \\ 0 \\ 0 \\ p_i c (E_i - E_p) \end{pmatrix} = \frac{1}{W} \begin{pmatrix} E_i E_p - p_i^2 c^2 \\ 0 \\ 0 \\ p_i c E_{0c} \end{pmatrix}$$

$$\begin{pmatrix} E'_c \\ 0 \\ 0 \\ p'_c c \end{pmatrix} = \frac{1}{W} \begin{pmatrix} E_i & 0 & 0 & -p_i c \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 0 \\ -p_i c & 0 & 0 & E_i \end{pmatrix} \begin{pmatrix} E_{c0} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{W} \begin{pmatrix} E_i E_{0c} \\ 0 \\ 0 \\ -p_i c E_{0c} \end{pmatrix}$$

(f) If an elastic collision occurs, then the total energy-momentum of the system is the same before and after the collision. In the zero-momentum system, the energy-momentum four vectors of the particles (the proton and carbon nucleus) after the collision can be found very easily. Describe in words what happens when elastic scattering occurs in the zero-momentum frame.

The energy of each particle remains the same before and after the collision. This means that the magnitude of the momentum of each particle must also be the same before and after the collision. In order for the total momentum to be conserved, the two particles must go off back to back so that the momentum remains zero. Thus, each particle changes direction, but nothing else changes in the collision process.

(g) Assume that the scattering angle of the proton is θ in the zero-momentum frame. Find the scattering angle in the original lab frame. The scattering angle is measured as the angle between the original direction of the proton (the z direction) and the scattering direction. For simplicity, assume that the proton remains in the x - z plane after the collision. Let a \sim represent the lab quantities after the collision.

$$\begin{pmatrix} \tilde{E}_p \\ \tilde{p}_p c \sin \theta_L \\ 0 \\ \tilde{p}_p c \cos \theta_L \end{pmatrix} = \frac{1}{W} \begin{pmatrix} E_i & 0 & 0 & +p_i c \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 0 \\ +p_i c & 0 & 0 & E_i \end{pmatrix} \begin{pmatrix} E'_p \\ p'_p c \sin \theta \\ 0 \\ p'_p c \cos \theta \end{pmatrix} = \frac{1}{W} \begin{pmatrix} E_i E'_p + p_i p'_p c^2 \cos \theta \\ W p'_p c \sin \theta \\ 0 \\ p_i c E'_p + E_i p'_p c \cos \theta \end{pmatrix}$$

$$\tan \theta_L = \frac{\sin \theta_L}{\cos \theta_L} = \frac{\tilde{p}_p c \sin \theta_L}{\tilde{p}_p c \cos \theta_L} = \frac{W p'_p c \sin \theta}{p_i c E'_p + E_i p'_p c \cos \theta}$$

2. A proton is traveling in the $+z$ direction at a speed of 1.30×10^8 m/s as measured by an observer in S . Find its velocity in a frame moving at a speed of 2.40×10^8 m/s in the $+z$ direction with respect to S . Do the problem in terms of energy-momentum four-vectors.

This time, we need only to do a simple boost into the S' frame:

$$\begin{pmatrix} E'_p \\ 0 \\ 0 \\ p'_p c \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E_p \\ 0 \\ 0 \\ p_p c \end{pmatrix} = \begin{pmatrix} \gamma E_p - \beta\gamma p_p c \\ 0 \\ 0 \\ \gamma p_p c - \beta\gamma E_p \end{pmatrix}$$

$$\beta' = \frac{p'_p c}{E'_p} = \frac{\gamma p_p c - \beta\gamma E_p}{\gamma E_p - \beta\gamma p_p c}$$

$$\beta' = \frac{\gamma\beta_p \gamma_p - \beta\gamma\gamma_p}{\gamma\gamma_p - \beta\gamma\beta_p \gamma_p}$$

$$\beta' = \frac{\beta_p - \beta}{1 - \beta\beta_p}$$

$$\beta' = \frac{1.30/3.00 - 2.40/3.00}{1 - (1.30/3.00)(2.40/3.00)} = -0.561$$

$$v' = 1.68 \times 10^8 \text{ m/s}$$

3. If an object is receding from the earth at half the speed of light, at what wavelength would an astronomer view light emitted at a wavelength of 400 nm?

Let the rest frame of the astronomical object be S . In this frame we know the wavelength and we know that $E=hf=hc/\lambda$. Let the earth, S' , move in the $+z$ direction with respect to S . Light from the object reaching the earth is also moving in the $+z$ direction. Hence:

$$\begin{pmatrix} E' \\ 0 \\ 0 \\ E' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix} = \begin{pmatrix} \gamma E - \beta\gamma E \\ 0 \\ 0 \\ \gamma E - \beta\gamma E \end{pmatrix}$$

$$hf' = hf\gamma(1 - \beta)$$

$$\frac{c}{\lambda'} = \frac{c}{\lambda} \gamma(1 - \beta)$$

$$\lambda' = \frac{\lambda}{\gamma(1 - \beta)} = \frac{400 \text{ nm}}{1.15 \times 0.5} = 692 \text{ nm}$$

This means that light emitted at the very blue end of the visible spectrum is seen at the very red end of the spectrum.

Homework 5–6

1. A bus 10.0 m long travels at 2.8×10^8 m/s. It passes through a garage with doors on the front and back. The garage is 8.00 m long.

(a) Can both doors of the garage be closed at the same time with the bus inside? Answer this from the perspective of a person in the garage and from a person on the bus. (Assume, of course, that you can open and close the doors very fast.) Is this contradictory?

$\beta = 0.933$ and $\gamma = 2.79$. To an observer in the garage, the bus appears to be only 3.58 m long and the bus can fit easily in the garage. To an observer on the bus, the garage appears to be only 2.87 m long, so the bus cannot fit inside. The seeming contradiction arises from the fact that simultaneity is relative. While the person in the garage sees both doors close at the same time, a person on the bus sees the front door close first and the back door close later.

(b) Work through the mathematical details to prove your argument. Define two events and determine the space-time coordinates of the events in both reference frames.

Let S be the garage frame and S' be the bus frame. Let S' move with velocity v in the $+z$ direction with respect to S .

In S :

Event 1: The left (back) door closes just after the bus enters the garage.

Event 2: The right (front) door closes simultaneously with the left door.

Thus:

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 8\text{m} \end{pmatrix}$$

$$\mathbf{x}'_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}'_2 = \begin{pmatrix} ct'_2 \\ 0 \\ 0 \\ z'_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 8\text{m} \end{pmatrix} = \begin{pmatrix} -8\text{m}\beta\gamma \\ 0 \\ 0 \\ 8\text{m}\gamma \end{pmatrix} = \begin{pmatrix} -21.4\text{m} \\ 0 \\ 0 \\ 22.9\text{m} \end{pmatrix}$$

Thus the observer on the bus sees the front door close $-21.4 \text{ m} / c = -71 \text{ ns}$ before the back door closes.

2. The rest energy of a π^0 particle is 135.0 MeV. A π^0 decays into two γ -ray photons with an average life-time of 1.2×10^{-16} s. (The half-life is 8.3×10^{-17} s.)

(a) Can two γ -rays have different energies in the rest frame of the π^0 ? Explain.

No, because they must come out back-to-back with the same momenta in order to conserve momentum.

(b) What are the energies of the two γ -rays as measured in the π^0 rest frame?

Each has half the rest energy of the π^0 , 67.5 MeV.

(c) In a laboratory, a beam of π^0 is produced. γ -rays from the π^0 decay are detected at angles of $\pm 22.0^\circ$ with respect to

the beam direction. What energy does each γ -ray have?

Let us call the beam direction the z direction. From the symmetry of the γ -rays detected, we can conclude that in the rest frame of the π^0 , the γ -rays were emitted in a direction perpendicular to z . We can call this the x direction. We can construct a photon energy-momentum four-vector in the π^0 rest frame, and then boost it to the lab frame. In the lab, the beam is moving in the z direction, so the boost must be in the $-z$ direction. Thus, calling the lab frame the S' frame:

$$\begin{pmatrix} E' \\ E' \sin\theta \\ 0 \\ E' \cos\theta \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ E \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma E \\ E \\ 0 \\ \beta\gamma E \end{pmatrix}$$

From this, we can conclude that

$$\begin{aligned} \tan\theta &= \frac{1}{\beta\gamma} \\ \beta^2\gamma^2 &= \cot^2\theta \\ \gamma^2 - 1 &= \cot^2\theta \\ \gamma^2 &= 1 + \cot^2\theta = \csc^2\theta \\ \gamma &= \frac{1}{\sin\theta} = 2.67 \end{aligned}$$

The energy is $E\gamma = 180 \text{ MeV}$.

(d) What is the kinetic energy of the π^0 in the lab frame?

$$K = 360 \text{ MeV} - 135 \text{ MeV} = 225 \text{ MeV}.$$

(e) What is the average distance such a π^0 would travel before decaying?

The average lifetime of the π^0 is its average rest lifetime times γ , $3.2 \times 10^{-16} \text{ s}$. Now we find the velocity, and then the distance:

$$\begin{aligned} \beta &= \sqrt{1 - \frac{1}{\gamma^2}} = 0.927 \\ d &= \beta ct = 8.9 \times 10^{-8} \text{ m}. \end{aligned}$$

Homework 5-7

1. A 32.0 m airplane is flying at Mach 2. By how much is its length contracted as measured by an observer on the ground?

The velocity is about 660 m/s, so $\beta = 2.2 \times 10^{-6}$, $\gamma = 1.00$. Since this doesn't tell us anything, we need to find γ by a Taylor series expansion:

$$\gamma = (1 - \beta^2)^{-1/2} \approx 1 + \frac{1}{2}\beta^2 = 1 + 2.42 \times 10^{-12}$$

The difference in length is then

$$2.42 \times 10^{-12} \times 32.0 \text{ m} = 7.74 \times 10^{-11} \text{ m}.$$

2. The "twin paradox" has often been used to refute relativity. In this paradox, two twins are initially on the earth

together. One twin goes off in a rapidly moving spaceship where her clocks tick slowly. After many years, she returns and finds herself younger than her twin. Since everything is relative, shouldn't each twin see the other one as younger?

No, the traveling twin has to accelerate, so the two twins are not in equivalent inertial frames.

Assume that the traveling twin can accelerate instantaneously to $\beta = 0.8$, then turn around and return to the earth at a similar speed, and then accelerate instantaneously to stop on the earth. If the round trip takes ten years as measured on the earth, by what amount would the traveling twin have aged?

Letting τ denote the time in the rest frame of the traveling twin,

$$\Delta t = \gamma \Delta \tau, \quad \Delta \tau = \frac{\Delta t}{\gamma} = 6 \text{ years.}$$

3. 1.00 kg and 1.00 kg of antimatter annihilate each other completely. How much energy is produced?

$$E_0 = mc^2 = 1.80 \times 10^{17} \text{ J}$$

Homework 5–8

1. In assignment 5–7, we introduced the “twin paradox.” Let us assume that the traveling twin experiences a constant force, F , as measured in the space ship. The force is directed away from the earth.

(a) Consider the space ship at one instant in time so it is instantaneously in an inertial reference frame S . Find the force as measured on the earth, using the relationship

$$\mathbf{f} = \frac{d\mathbf{E}}{d\tau} = \gamma \frac{d\mathbf{E}}{dw} = \gamma \begin{pmatrix} \vec{\beta} \cdot \mathbf{F} \\ F_x \\ F_y \\ F_z \end{pmatrix}.$$

In the space ship's frame, S , the twin is at rest. Thus
$$\mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F \end{pmatrix}.$$

The earth, S' , moves in the $-z$ direction with respect to S . On the earth, this becomes

$$\begin{pmatrix} \gamma & 0 & 0 & +\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ F \end{pmatrix} = \begin{pmatrix} \beta\gamma F \\ 0 \\ 0 \\ \gamma F \end{pmatrix}$$

$$F' = \frac{1}{\gamma} \gamma F = F$$

Note that the two γ s are the same. Thus the force is the same in the earth frame as in the frame of the space ship.

(b) Given that $F = dp/dt$, find γ as a function of time.

$$\begin{aligned}\frac{dp}{dt} &= F, & p &= Ft, & pc &= Fct \\ E_0\beta\gamma &= Fct \\ \beta^2\gamma^2 &= \gamma^2 - 1 = \frac{F^2c^2}{E_0^2}t^2 \\ \gamma &= \sqrt{1 + \frac{F^2c^2}{E_0^2}t^2}\end{aligned}$$

(c) (Optional.) Find a relationship between t and τ . (*Hint: $dt = \gamma d\tau$*)

$$\begin{aligned}\frac{dt}{d\tau} &= \sqrt{1 + \frac{F^2c^2}{E_0^2}t^2} \\ \int d\tau &= \int \frac{dt}{\sqrt{1 + \frac{F^2c^2}{E_0^2}t^2}} \\ \tau &= \frac{E_0}{Fc} \ln \left(\frac{Fc}{E_0}t + \sqrt{1 + \frac{F^2c^2}{E_0^2}t^2} \right)\end{aligned}$$