Physics 121
Class 17

What is the moment of inertia of this disk?

• Saturday's Quiz

Reminders

• Quiz #6 is due Monday!

Last Time

• We discussed places where the moment of inertia is important
• We learned how to obtain moments of inertia from tables
• We integrated to find mass, center of mass, and moment of inertia.
Today

- More on moments of inertia and integration
- Angular momentum and torque as vectors
- Representingcw and ccw as vector directions
- Cross products
- Sample problems

Mass by Integration

Cartesian Coordinates

The small bit of mass in a region is
\[ dm = \rho \, dV \]

and in Cartesian coordinates, we can take
\[ dV = dx \, dy \, dz \]

Then all we need to do is add up all the dm.

\[ M = \int dm = \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} \rho \, dx \, dy \, dz = \int_{z=0}^{c} \int_{y=0}^{b} \rho \, dy \, dz = \int_{z=0}^{c} \rho \, ab \, dz = \rho abc \]
Cartesian Coordinates

Let's assume $\rho = Ax^2$ is no longer constant.

\[
M = \int dm = \int_{-b}^{b} \int_{-a}^{a} \int_{0}^{\infty} A x^2 \, dx \, dy \, dz
= \int_{-b}^{b} \int_{-a}^{a} \frac{1}{3} A x^3 \, dy \, dz
= \int_{-b}^{b} \frac{1}{3} A a^3 b \, dz
= \frac{1}{3} A a^3 b c
\]

Cylindrical Coordinates

\[
V = \int dV = \int 2\pi r \rho (r) dr
d\tau = 2\pi \rho L(r) dr
m = \int dm = \int \rho dV = \int \rho (r) 2\pi \rho L(r) dr
\]

Cylindrical Coordinates

Let's assume $\rho = \rho(r)$ only. We slice the object into cylindrical shells. Each shell has length $L$ which may be a function of $r$. The object has an outside radius of $a$.

\[
M = \int dm = 2\pi \int_0^a \rho (r) L(r) \, dr
\]

You don't need to memorize this, but you should be able to use it. This is more important than the Cartesian formula.

Mass of a Spheroid

Let's let $\rho$ be a constant.

\[
M = \int dm = 2\pi \int_0^a \rho 2a \sqrt{1 - r^2 / b^2} \, dr
= \frac{4}{3} \pi \rho ab^2
\]

Note: If $a=b$, it's a sphere...
Center of Mass - Point Masses

If we have several point masses, the center of mass can be found with the equation:

\[ \bar{x} = \frac{\sum m_i x_i}{\sum m_i} \]

There are similar equations for y and z.

See-saw

Children of masses M and m sit on the ends of see-saw of length 1, and mass m. Where would you put the fulcrum of the see-saw so the children would balance?

CM by Integration
Let's assume $\rho$ is not a constant.

$$x_{cm} = \frac{1}{M} \int x \, dm = \frac{1}{M} \int \left[ \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} x \rho \, dx \, dy \right] \, dz$$

Let $\rho = Ax^2$

$$x_{cm} = \frac{1}{M} \int \left[ \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} x Ax^2 \, dx \, dy \right] \, dz$$

$$= \frac{3}{Aa^2} \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} 4Ax^2 dy \, dz$$

$$= \frac{3}{Aa^2} \int_{y_{min}}^{y_{max}} 4Aa^2 dy \, dz$$

$$= \frac{3}{Aa^2} \frac{4}{4Aa^2} = \frac{3a}{4}$$

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By Integration

We find the moment of inertia around the $z$ axis. There $dI = x^2 + y^2$

$$I_z = \int \int \int (x^2 + y^2) \rho \, dx \, dy \, dz$$

How is the block positioned if we use these limits of integration?
Cartesian Coordinates

Let the density be constant.

\[ I_z = \int d^2 y\, dm = \rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2)\, dy\, dx \int dz \]

\[ = \rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{a^2}{3} + ayb \right)\, dy\, dx \int dz \]

\[ = \rho \left( \frac{a^2}{3} b + \frac{ab}{3} \right) dz \]

\[ = \rho \left( \frac{a^2}{3} b + \frac{ab}{3} \right) = \frac{M}{3}(a^2 + b^2) \]

as \( M = \rho ab c \)

I of a Spheroid

Let's let \( \rho = \) be a constant. Find I about the long axis.

\[ I_{xx} = \int r^2 \, dm = 2\pi \int_{0}^{\infty} r^2 \rho a^2 \sqrt{1 - r^2/b^2} \, dr \]

\[ = 2\pi \int_{0}^{\infty} r^2 \rho a^2 \sqrt{1 - r^2/b^2} \, dr \]

\[ = \frac{8}{15} \pi \rho ab^4 \]

\[ M = \frac{4}{3} \pi \rho ab^4 \]

\[ \frac{2}{3} Mb^2 \]

\[ \frac{2}{3} \frac{M}{b^2} \]

\[ \frac{2}{3} \frac{M}{b^2} \]

\[ L(r) = 2\pi r \]

Note: If \( a = b \), it's a sphere...

\[ L = 2\pi \sqrt{1 - r^2/b^2} \]

Compare \( p \) and \( L \)

Linear momentum is

\[ p = m v \]

Angular momentum is

\[ L = I\omega \]
Compare $p$ and $L$

Momentum is a vector. Its direction is the direction of the velocity.

$$\vec{p} = mv$$

Angular momentum is a vector, but what is its direction?

Direction of $L$

For now we will take the angular momentum about the axis of rotation.

Direction of $L$

A stationary wheel is rotating about an axis. What can we say about the direction of rotation?

Just whether the rotation is clockwise or counter-clockwise.
**CW - CCW Convention**

The convention is to assign a direction to CW or CCW rotation along the axis of rotation. The choices are only left or right along the axis?

![Diagram](image1)

**L for Moving Objects**

We apply the same idea to the direction of angular momentum for a moving object. The direction just differentiates CW from CCW rotation.

1. Choose an origin for a coordinate system.
2. The vector \( \mathbf{r} \) goes from the origin to the object.
3. The vectors \( \mathbf{r} \) and \( \mathbf{v} \) form the “rotation plane.”
4. Think of \( \mathbf{v} \) as a rotation - put the fingers of your right hand in the direction of \( \mathbf{v} \).
5. The direction of the angular momentum is perpendicular to the rotation plane in the direction of your thumb.
**Cross Product**

The angular momentum and torque of an object are defined in terms of the "cross product."

\[ \vec{L} = \vec{r} \times \vec{p} \]
\[ \vec{\tau} = \vec{r} \times \vec{F} \]

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**Definition**

\[ \vec{A} \times \vec{B} = AB \sin \theta \ \hat{n} \]

- The magnitude of the cross (or vector) product is \( AB \sin \theta \).
- The direction is given by the right-hand rule and is always perpendicular to \( \vec{A} \) and to \( \vec{B} \).
- If vectors are parallel or antiparallel, the cross product is zero.
- If vectors are perpendicular, the cross product is maximum.

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**Useful Rules**

\[ \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \]
\[ \vec{A} \times \vec{A} = 0 \]
\[ \hat{i} \times \hat{j} = \hat{k} \]
\[ \hat{j} \times \hat{k} = \hat{i} \]
\[ \hat{k} \times \hat{i} = \hat{j} \]
RHR

\[ \vec{A} \times \vec{B} = AB \sin \theta \, \hat{n} \]

Remember the cross product is always perpendicular to the plane of \( \vec{A} \) and \( \vec{B} \). There are only two such directions. The RHR chooses between the two perpendicular directions.

The cross product can only be toward you or away from you!

RHR - v.1

\[ \vec{A} \times \vec{B} = AB \sin \theta \, \hat{n} \]

Place your fingers of your right hand in the direction of \( \vec{A} \) in such a way that they rotate into \( \vec{B} \). The cross product is in the direction of your thumb.

The cross product is toward you.

RHR - v.2

\[ \vec{A} \times \vec{B} = AB \sin \theta \, \hat{n} \]

Point the index finger of your right hand in the direction of \( \vec{A} \) and your middle finger in the direction of \( \vec{B} \). The cross product is in the direction of your thumb.

The cross product is toward you.

RHR - v.3

\[ \vec{A} \times \vec{B} = AB \sin \theta \, \hat{n} \]

Put your thumb in the direction of \( \vec{A} \). Put your other fingers in the direction of \( \vec{B} \) with your hand out straight. (Don’t bend your fingers.) The cross product outward from your palm.

The cross product is toward you.
**RHR - v.4**

\[ \vec{A} \times \vec{B} = AB \sin \theta \, \hat{n} \]

Think of a screw at the origin (tails) of the vectors. Turn the screw from \(A\) to \(B\). The cross product is in the direction the screw moves (in or out).

**Inclined Plane**

A ball is rolling down an inclined plane. What is the direction of \(L\) about the axis of rotation? What is the direction of \(\tau\) about the axis?

**Inclined Plane**

A ball is rolling down an inclined plane. Find an expression for \(\vec{\tau} = \frac{d\vec{L}}{dt}\) works in this case.
Slide 45 - Conservation of $L$

Conservation of $L$

Slide 46 - Slide 46

When $L$ is conserved

Angular momentum is conserved (it does not change in time) when there is no torque.

$$\ddot{\vec{L}} = \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} \text{ is constant}$$

Slide 47 - Slide 47

Demonstrations

What happens when there is no torque and $I$ increases? decreases?

What happens when I stand on a platform holding a rotating bicycle tire with $L$ up and the tire is tipped so $L$ is down?

Slide 48 - End

End