von Neumann Stability Analysis
The Diffusion Equation

In order to determine the Courant-Friedrichs-Levy condition for the stability of an explicit solution of a PDE you can use the von Neumann stability analysis. To do this you assume that the solution is of the form $T_j^n = \xi^n e^{ikjh}$ where ξ represents the time dependence of the solution and the exponential represents the spatial dependence. It is assumed that the coefficients in the equation are so slowly varying that they may be considered constant in both space and time. Substituting this into the finite difference formula:

$$\frac{T_j^{n+1} - T_j^n}{\tau} = D\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{h^2}$$

$$\frac{\xi^{n+1}e^{ikjh} - \xi^n e^{ikjh}}{\tau} = D\frac{\xi^n e^{ik(j+1)h} - 2\xi^n e^{ikjh} + \xi^n e^{ik(j-1)h}}{h^2}$$

$$\frac{\xi - 1}{\tau} = D\frac{e^{ikh} - 2 + e^{-ikh}}{h^2}$$

$$\xi - 1 = \frac{2D\tau}{h^2}(\cos(kh) - 1)$$

$$\xi = 1 + \frac{2D\tau}{h^2}(\cos(kh) - 1)$$

If $\cos(kh) = 1$ then $\xi = 1$. If $\cos(kh) = -1$ (the second term is as negative as possible) then $|\xi| \leq 1$ only if $2D\tau/h^2 \leq 1$ or $\tau \leq h^2/(2D)$ which is the condition used in Exercise 8.

If we look at the equation with centered time derivatives

$$\frac{T_j^{n+1} - T_j^{n-1}}{2\tau} = D\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{h^2}$$

$$\frac{\xi^{n+1}e^{ikjh} - \xi^{n-1}e^{ikjh}}{2\tau} = D\frac{\xi^n e^{ik(j+1)h} - 2\xi^n e^{ikjh} + \xi^n e^{ik(j-1)h}}{h^2}$$

$$\frac{\xi^2 - 1}{2\tau} = D\xi\frac{e^{ikh} - 2 + e^{-ikh}}{h^2}$$

$$\xi^2 - \xi\frac{4D\tau}{h^2}(\cos(kh) - 1) - 1 = 0$$

$$\xi = \frac{2D\tau}{h^2}(cos(kh) - 1) \pm \sqrt{\frac{4D^2\tau^2}{h^4}[cos(kh) - 1]^2 + 1}$$

if $\cos(kh) = 1$ then $\xi = 1$. If $\cos(kh) = -1$ then

$$\xi = -\frac{4D\tau}{h^2} \pm \sqrt{\frac{16D^2\tau^2}{h^4} + 1}$$

Since we are looking for the condition $|\xi| \leq 1$, the worst case is going to be when we use the negative sign on the square root. Since the square root is always greater than 1 and both terms are negative then the magnitude of ξ will always be greater than one no matter how small we choose τ to be.

And now to look at the stability of the Crank-Nicholson scheme. From the initial finite differencing

$$\frac{T_j^{n+1} - T_j^n}{\tau} = D\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{h^2}$$

we wish to move the spatial derivatives to time $n + 1/2$ by using the averages:

$$T_j^{n+1} - T_j^n = \frac{D\tau}{h^2}\left(\frac{T_{j+1}^{n+1} + T_{j+1}^n}{2} - 2\frac{T_j^{n+1} + T_j^n}{2} + \frac{T_{j-1}^{n+1} + T_{j-1}^n}{2}\right)$$

Substituting $T_j^n = \xi^n e^{ikjh}$

$$\begin{aligned}\xi^{n+1}e^{ikjh} - \xi^n e^{ikjh} = {} & \tfrac{D\tau}{2h^2}(\xi^{n+1}e^{ik(j+1)h} + \xi^n e^{ik(j+1)h} - 2\xi^{n+1}e^{ikjh} \\ & -2\xi^n e^{ikjh} + \xi^{n+1}e^{ik(j-1)h} + \xi^n e^{ik(j-1)h})\end{aligned}$$

$$\xi - 1 = \frac{D\tau}{2h^2}(\xi e^{ikh} + e^{ikh} - 2\xi - 2 + \xi e^{-ikh} + e^{-ikh})$$

$$\xi = 1 - \frac{D\tau}{h^2}(\xi + 1)[1 - \cos(kh)]$$

$$\xi = \frac{1 - \frac{D\tau}{h^2}[1 - \cos(kh)]}{1 + \frac{D\tau}{h^2}[1 - \cos(kh)]}$$

It is easy to see that the magnitude of the right-hand side of this equation is always less than or equal to one no matter what the value of τ. This algorithm is inherently stable.