HANDLING EXPERIMENTAL UNCERTAINTY

HOW TO DETERMINE UNCERTAINTY

In reporting experimental results, data are usually listed with a numerical value for the uncertainty. For example, the best value of the mass of an electron currently is $(5.48579902 \pm 0.00000013) \times 10^{-4}$ atomic mass units.

For the purposes of most analyses, and in particular for the purposes of this class, the rules for assigning uncertainty follow:

1. For statistical uncertainty, use one standard deviation (unless otherwise specified).

2. For instrumental uncertainty, estimate the uncertainty as well as you can. Although this uncertainty isn't strictly Gaussian, it is usually treated as if it were Gaussian in subsequent error analysis.

3. For known systematic errors, the data should be appropriately corrected and no systematic uncertainty quoted.

PROPAGATION OF ERRORS: QUICK FORMULA

$$(\Delta f)^2 = \left( \frac{\partial f}{\partial a} \right)^2 (\Delta a)^2 + \left( \frac{\partial f}{\partial b} \right)^2 (\Delta b)^2 + \left( \frac{\partial f}{\partial c} \right)^2 (\Delta c)^2 + \ldots.$$  

PROPAGATION OF ERRORS (SEE LAB MANUAL)

You may often desire to calculate a quantity mathematically from other values that have been obtained experimentally. Each of these experimental values will have associated uncertainties, and you must consider the effect of these uncertainties on the calculated quantity. For example, the calculated density $\rho$ of a circular cylindrical object is given by

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{\pi R^2 L}. \quad (1)$$

But $M$, $R$, and $L$ might be measured values with uncertainties $\Delta M$, $\Delta R$, and $\Delta L$. What then is the uncertainty $\Delta \rho$ in $\rho$?

For statistical or random errors, a detailed analysis allows you to determine the uncertainty
in a calculated quantity in terms of the uncertainties in the measured quantities. If we take a
general function \( f \) of the variables \( a, b, c, \ldots \), it can be shown that the uncertainty in \( f \) can be
calculated by the relationship:

\[
(\Delta f)^2 = \left( \frac{\partial f}{\partial a} \right)^2 (\Delta a)^2 + \left( \frac{\partial f}{\partial b} \right)^2 (\Delta b)^2 + \left( \frac{\partial f}{\partial c} \right)^2 (\Delta c)^2 + \ldots.
\]  

(2)

Again, this relationship assumes all errors are random and are Gaussian in nature. We see that
there are two essential contributions from each variable to the uncertainty in \( f \): the partial
derivative of \( f \) with respect to the variable, and the uncertainty in the variable. Clearly, the greater
the uncertainty in a given variable, the greater will be the uncertainty in the derived quantity, \( f \).
Also if \( f \) depends strongly on a given variable, the uncertainty in that variable will be more
important to the uncertainty in \( f \); hence, the partial derivative appears as a multiplicative factor in
each term. Furthermore, note that the uncertainties add in “quadrature” like the components of a
vector or the sides of a right triangle.

In case you have never before seen a partial derivative, the concept is really quite simple.
The function \( f \) depends on several variables; however, in each term of equation (2) we are only
interested in one variable at a time. Hence, when we evaluate \( \partial f / \partial a \), we treat \( b, c, \ldots \) as constants
while taking the derivative of \( f \) with respect to \( a \).

This general formula can be used for any relationship, such as the one for density above.
There

\[
\frac{\partial \rho}{\partial M} = \frac{1}{\pi R^2 L} = \frac{\rho}{M},
\]

\[
\frac{\partial \rho}{\partial L} = \frac{-M}{\pi R^2 L^2} = -\frac{\rho}{L},
\]

\[
\frac{\partial \rho}{\partial R} = \frac{-2M}{\pi R^3 L} = -\frac{2\rho}{R}.
\]

Hence

\[
\left( \frac{\Delta \rho}{\rho} \right)^2 = \left( \frac{\Delta M}{M} \right)^2 + \left( \frac{\Delta L}{L} \right)^2 + 4 \left( \frac{\Delta R}{R} \right)^2.
\]

Although equation (6) is general, the following expressions are often useful:

1. If \( f = \pm a \pm b \pm c \)
\[(\Delta f)^2 = (\Delta a)^2 + (\Delta b)^2 + (\Delta c)^2 \]  \hspace{1cm} (3)

2. If \( f = abc \) or \( f = albc \), etc.

\[
\left( \frac{\Delta f}{f} \right)^2 = \left( \frac{\Delta a}{a} \right)^2 + \left( \frac{\Delta b}{b} \right)^2 + \left( \frac{\Delta c}{c} \right)^2
\]  \hspace{1cm} (4)

The quantity \( \Delta f / f \) is termed the fractional or relative error in \( f \).