Preface

**Course Description:** Introduction to physical measurement and analysis, optics, sensors, actuators, and computer-based data acquisition.

**Prerequisites:** Physics 121, 140.

This manual accompanies a hands-on course designed to help students become familiar with a variety of laboratory instruments and techniques. Students will be expected to carefully record observations and develop experimental ‘common sense’ and physical intuition. Exercises include geometric and wave optics, Fourier transforms, AC impedances (electrical, mechanical and acoustical), time and frequency-dependent transducer responses, data analysis (e.g. statistics, error estimation and propagation, curve fitting), and computer-assisted data acquisition and experiment control (with Labview software).

**Lab Reports:** Students will be required to document their work in a personal lab notebook, which will be regularly evaluated for grading purposes. They are expected to create an accurate and thorough record of each experiment as it is conducted, including a record of motivations, questions, insights and interpretations. Graphs and printouts of computer code can be taped into the notebook when needed. If a section is found to be in error, it is good practice simply to place a big X over the section, still leaving it readable (in case some of the information is later found to be valuable). While students work together in teams (usually pairs) at each lab station, students will be graded individually based on the report that they prepare in their lab notebook. Notebooks are submitted at the end of each lab period for grading and returned by the start of the next period.

Each lab report should include the following headings:

**Quiz**
Students are expected to read the lab material before each lab session. The quiz questions are designed to increase depth of understanding and will help prepare students for the final exam. Students may update their quiz answers during the lab period.

**Procedures, Data and Observations**
This section is to be written entirely while performing the experiments. This record, which is the heart of the report, is a step-by-step description of experimental procedures and the data that is collected, as well as an explanation of
calculations. It should include sketches, tables, and graphs to make the record as clear as possible. Students are encouraged to insert numerous real-time observations, insights, ideas, and reactions during the course of the experiment (e.g. “That didn't turn out as expected – why?”).

Discussion and Conclusion
Here students are encouraged to 'think in the notebook' about the results of the completed experiment, interpret the outcome and its significance, and concisely state the conclusions of experiments.
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Lab 1

Geometric and Wave Optics

Optical phenomena are an important part of many experiments, and often require only simple equipment. In this lab, you will explore image formation by lenses and mirrors as well as some of the physical properties of light, namely wavelength and polarization.

1.1 Ray Optics

Become familiar with the image-forming properties of thin lenses and concave/convex mirrors. An excellent tutorial is available at the Hyperphysics website created by Georgia State University: http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html. (Alternatively, you may wish to refer to the Physics 123 textbook.) During class, please spend a while in the section on Light and Vision at the website. Note the links to topics such as Reflection, Lenses, Lens Equation and Mirrors.

It is important to gain a solid appreciation for the image-formation equation

\[
\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}
\]

which relates the location of an image \(d_i\) to the location of an original object \(d_o\). Of course, this is controlled by the focal length \(f\) of the lens. The lens equation also works with spherical mirrors, where \(f\) is equal to half the radius of curvature of the mirror.

When an image is formed, it means that the many rays of light emitted from each point on an object converge to a corresponding point at the image. It is often helpful to draw a ray diagram such as shown in Fig. 1.1. Note that a ray that travels parallel to the axis before the lens goes through the focus after the lens (assuming positive focal length). A ray that goes through the focus before the lens travels parallel to the axis afterwards. A ray that goes through the center of a lens is always un-deflected.

Note that \(d_i\) and \(d_o\) (as well as \(f\)) can be either positive or negative, depending on which side of the lens they are located on. If an image cannot be displayed on
a screen (i.e. it must be observed by looking into the lens or mirror), we say that
the image is virtual. This happens when \( d_i \) is negative.

**Sign conventions for lenses and mirrors**

**Focal length:** convex lens (+), concave lens (−), convex mirror (+), concave mirror (+)

**Object:** (+) is always the source side (upstream).

**Real Image:** far side of lens (downstream) (+); source side of mirror (downstream after reflection) (+).

**Virtual Image:** source side of lens (upstream) (−); behind mirror surface (−).

Also spend some time exploring a variety of lens and mirror focal lengths (positive and negative) using the Optics Bench Physlet resource created by Davidson College: [http://webphysics.davidson.edu/Applets/optics4/default.html](http://webphysics.davidson.edu/Applets/optics4/default.html). Insert objects and find the relative sizes and locations of the resulting images. Insert point sources and parallel light sources (i.e. beams) and see how lenses and mirrors affect them.

The size of an image \( h_i \) is not necessarily the same as the size of the object \( h_o \). The image is either enlarged or shrunken in proportion to its distance from the lens. The magnification is the ratio of the image size to the size of the original object, and it is given by

\[
M \equiv \frac{h_i}{h_o} = -\frac{d_i}{d_o} \quad (1.2)
\]

If it is negative, it means that the image is ‘upside down’.

### 1.2 Compound Lens System: Telescope

For instruments containing more than one lens, angular magnification is also an important concept – by magnifying the angular width of an object, a telescope or microscope makes an object appear larger. For small angles, \( \theta \approx \tan \theta = h/d \), so that angular magnification can be computed as \( \theta_i/\theta_o = (h_i/d_i)/(h_o/d_o) \). In the case of a telescope, comprised of two lenses separated by the sum of their focal lengths, the angular magnification becomes

\[
m_\theta \equiv \frac{\theta_i}{\theta_o} = -\frac{f_1}{f_2} \quad (1.3)
\]

### 1.3 Physical Optics

The wave optics portion of the lab (Part II) reveals that microwaves have much in common with visible light, while also permitting a direct measurement of wavelength. Because of their convenient wavelength range (millimeters to centimeters), working with microwaves allows one to build intuition in a way that isn't possible with visible electromagnetic radiation.
1.4 Equipment

Part A: desk lamp, optical bench with lens holders, lens and mirror set, large and small magnification scales. Part B: Microwave generator and detector with digital multimeter, microwave polarizer, microwave reflector with tape measure, curved microwave mirror, microwave prism.
Quiz

Q1.1 The focal length of the lens shown in Fig. 1.2 is
(a) positive.
(b) negative.
(c) concave.
(d) convex.

Q1.2 For the object and lens shown in Fig. 1.2, the image is
(a) real.
(b) virtual.
(c) inverted.
(d) all of the above.

Q1.3 The focal length of the lens above has a magnitude of 10 cm. The object is \( d_o = +20\text{cm} \) from the lens and has a height of \( h_o = 5\text{cm} \). Calculate the location \( d_i \) and the height \( h_i \) of the image (both signs and magnitudes).

Q1.4 Explain the difference between lateral and angular magnification.
Exercises

A. Geometric Optics.

L1.1 Measure the focal length of a positive lens or mirror. The focal length of a positive mirror or lens is the distance from the lens center or mirror surface to the point where parallel light rays converge. You can obtain an approximately parallel beam of light rays by using a distant lamp as a source.

L1.2 Experimentally verify the lens and lateral-magnification equations for a positive lens.

L1.3 With a positive lens of focal length $f$, determine the minimum possible distance between an object and its real image? Hint: use a lens with a fairly short focal length, and move the lens between the object and a screen to find an image at two locations of the lens. Then move the screen closer and repeat the procedure to experimentally determine the answer to the question. The theoretical limit is $4f$. Explain this result in terms of the lens equation.

L1.4 Create a simple telescope and demonstrate that its angular magnification is $m = f_{\text{obj}} / f_{\text{eye}}$. Here, $f_{\text{obj}}$ and $f_{\text{eye}}$ are the focal lengths of the objective and eyepiece lenses, respectively. A high angular magnification is obtained by using a short eyepiece focal length and a long objective focal length, and arranging the lenses so that they share a common focal point. Try using a few different lens combinations to read some small print from across the room. With a little practice you can simultaneously look through the telescope with one eye and past the telescope with the other and let your brain superimpose the images for visual comparison. The large scale on the wall should help.

B. Wave Optics.

L1.5 Explore the polarization state of the microwaves being emitted by the generator. A beam of electromagnetic radiation is said to be linearly polarized if the electric field of the beam has a well-defined direction. Your microwave detector measures the potential difference (i.e. voltage) between the ends of a small antenna, and responds most strongly when the antenna is parallel to the polarization direction.

(a) Vary the antenna orientation to determine the polarization direction. Do you find the microwave polarization to be transverse (i.e. perpendicular to the beam direction) or longitudinal (parallel to the beam direction)?

(b) Use a stand to fix the antenna parallel to the microwave polarization direction, insert the polarizer into the beam between the source and
the antenna, and vary the polarizer angle. To avoid interference, try to keep other metal objects out of the vicinity of the microwave beam path. Verify that the signal detected is proportional to \( \cos^2 \phi \), where \( \phi \) is the angle of the polarizer relative to the maximum intensity position. Can you explain the effect of the wires?

(c) Now rotate the antenna direction to be perpendicular to the microwave polarization and also perpendicular to the beam direction, which should yield a zero signal. Try slowly rotating the polarizer again. Describe and explain what you observe?

L1.6 Measure the wavelength of the microwave radiation. Set up a standing wave by reflecting the microwave beam from a metal sheet back towards the source. Place the metal sheet at least 60 cm from the source, and place the detector, which should be oriented parallel to the microwave polarization direction, close to the metal sheet. You should see a series of peaks in the detected microwave intensity when scanning the detector towards the source, where two adjacent peaks are separated by half a microwave wavelength. Mounting your detector on the optical bench may make it easier to maintain proper position and alignment while scanning. Adjusting the position of the metal sheet slightly may also improve the contrast of the peaks. Now measure the wavelength. To improve accuracy, you can measure the distance that spans five peaks and divide accordingly.
Lab 2

Spectroscopy

In this lab you will measure and quantify emission spectra from several different visible light sources.

2.1 Spectral Lines

In physics, we typically use the word *spectrum* to refer to the frequencies of a set of superposed sine waves. One can always think of a signal of some sort as the superposition of sinusoidal waves with different frequencies. *Spectroscopy* is the science of separating a signal into its spectral components in order to determine how much amplitude or intensity each sine wave contributes to an overall signal.

Almost any physical phenomenon (electromagnetic/optical, electrical, acoustical, mechanical, etc.) can be characterized in terms of a spectrum. Examples include transmissions from modern communications devices, the light from distant galaxies, the wobble of a planet due to the motions of its moons, the swaying of a large building during an earthquake, the sound from a high-quality audio system, the x-rays used for a medical exam, the molecules that give plants their colors, and the energies of fundamental particles produced in an accelerator. In each case, the spectral signature allows one to look ‘under the hood’ for insights that would be invisible to a naive observer. Spectral data can be used to probe the physical characteristics of a signal source or the response of a dynamical system to an incoming signal.

In this lab, we will investigate the (visible) spectral properties of common light sources (e.g. sunlight, incandescent bulbs, fluorescent lamps, LEDs). Some sources exhibit a continuous spectrum (e.g. a hot filament of the Sun) while others exhibit a few discrete *spectral lines* (e.g. electrically excited elemental gases). Fig. 2.1 shows a discrete spectrum of an elemental gas superposed on a continuous white-light spectrum.
2.2 Wavelength and Frequency

An optical spectrometer often employs a grating to separate different wavelengths of light into an image. Below is a summary of how wavelength and frequency are connected, as well as how they relate to other properties of photons.

**Definitions and relationships**

\[
\begin{align*}
  k &= \frac{2\pi}{\lambda} = 2\pi \tilde{k} \\
  \omega &= \frac{2\pi}{T} = 2\pi f \\
  c &= \lambda f = \frac{\omega}{k} = \frac{\lambda}{T} = \frac{f}{\tilde{k}} \\
  E &= h\omega \\
  p &= \hbar k
\end{align*}
\]

- \( f \) = cyclic frequency (cycles/s or Hz)
- \( \lambda \) = wavelength (m)
- \( \omega \) = angular frequency (radians/s)
- \( k \) = angular wavenumber (radians/m)
- \( T \) = period (s)
- \( \tilde{k} \) = cyclic wavenumber (cycles/m)
- \( E \) = energy (J or eV)
- \( p \) = momentum (kg·m/s)

The parameter \( c \) is the speed of light in vacuum \((3.0 \times 10^8 \text{ m/s})\) and \( \hbar \) is Planck’s constant over \( 2\pi \) \((1.055 \times 10^{-34} \text{ J·s} = 6.582 \times 10^{-16} \text{ eV·s})\). Because one can easily interconvert between any two of these quantities, any one of them could serve as a spectral parameter (i.e. the horizontal axis of a spectrum graph). If you know one, you know them all. In this lab, we will use wavelength as the primary spectral parameter.

2.3 Blackbody Spectra

Hot objects glow. In the latter 1800’s scientists discovered that the radiation emitted by hot objects as a function of frequency is approximately the same for all materials. The notion that all materials behave similarly led to the concept of an ideal blackbody radiator.\(^1\)

\(^1\)The name comes from the fact that objects that absorb light better are also able to radiate light the best. Light that falls upon an ideal blackbody is absorbed perfectly before the possibility of reemission.

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The Sun is a good example of a blackbody radiator. The light emitted from the Sun is associated with its surface temperature. As another example, a glowing tungsten filament in an ordinary light bulb may be reasonably described as a blackbody radiator. The structure of the material must be sufficiently complex (e.g., a solid or a dense plasmas) to enable absorption and emission at any wavelength. On the other hand, a diffuse atomic vapor can only absorb and emit wavelengths that correspond to allowed energy transitions within the atom, which is not well described by blackbody radiation.

Fig. 2.2 shows the spectrum of blackbody radiation. As the temperature increases, the peak of the curve shifts to shorter wavelengths. The peak is given by Wien’s Displacement Law

\[ \lambda = \frac{2.9 \times 10^6 \text{ nm} \cdot \text{K}}{T} \]  \hspace{1cm} (2.1)

### 2.4 Equipment

Hand-held spectrometer, digital spectrometer, Labview-enabled computer, Geisler-tube power supply and a set of Geisler tubes, white LED flashlight.

![Figure 2.2: Blackbody spectrum for different temperatures.](image-url)
Quiz

Q2.1  The wavelength and frequency of light can be selected independently. True or False?

Q2.2  Why do elemental gases have discrete rather than continuous spectra?

Q2.3  A red-light photon from a helium-neon laser has a wavelength of $\lambda = 632.816$ nm. Calculate each of the following properties of this photon:

(a) period $T$

(b) cyclic frequency $f$

(c) angular frequency $\omega$

(d) energy $E$ (in electron-volts, where $1 \text{ eV} = 1.609 \times 10^{-19} \text{ J}$)

(e) cyclic wave number

(f) angular wave number $k$

(g) momentum $p$

Q2.4  For this lab, to calibrate a spectrometer means to establish which correspondence?

(a) wavelength of light vs. detector pixel number

(b) arrival time of light vs. detector readout time

(c) intensity of light vs. detector response
Exercises

A. Qualitative observations

Please use the foot pedals provided to deliver power to the Geisler-tube power supply, as leaving it on continuously greatly shortens the tube lifetimes. Don't insert or remove Geisler tubes unless the power supply is switched off.

L2.1 Use the hand-held spectroscope to study the following light sources: sunlight (if you can find some) or an incandescent desk lamp, fluorescent room lights, mercury Geisler tube. Describe the most important characteristics of each spectrum and explain the physical phenomena underlying their differences. Tips: You are welcome to use books or online resources for extra information. Relevant keywords are “spectroscopy”, “black-body spectrum” and “fluorescent lamp”.

B. Calibration

Connect the digital spectrometer to a USB port on your computer. Use the wall transformer provided to deliver power to the spectrometer. Remove the rubber tips from both ends of the fiber optic and insert one end into the spectrometer. Support the other end of the fiber optic and aim it at a helium Geisler tube source from a distance of about 10 cm.

Find the “digitalspectrometer” program on your computer (there should be a shortcut to the Labs/Physics145 folder on the desktop). This program was written using the Labview software environment. The program should start automatically when you open it. The small red "stop sign" icon will stop it, and the adjacent black arrow icon will restart it. Before reaching the position-sensitive detector, the light is reflected off a finely-divided grating, which causes different wavelengths of light to be reflected in slightly different directions, thereby reaching different detector pixels.

L2.2 Click on “Acquire Single Spectrum” to collect an uncalibrated spectrum. The vertical axis of the graph is intensity, while the horizontal axis is measured in arbitrary units that correspond to the pixels of a position-sensitive CCD detector chip. If necessary, adjust the distance between the fiber optic and the Geisler tube to ensure that you have adequate signal without letting any of the peaks saturate the detector. Print a copy of the graph for your lab notebook – use scissors and scotch tape.

L2.3 You now need to teach the instrument how to convert pixel number into wavelength in nanometers. This process is called calibration. Click on “Calibrate Spectrometer” to open a separate calibration window, and then click “Take Spectrum” to collect another helium gas spectrum.
Note that wavelength is approximately proportional to detector pixel, so that wavelength increases to the right on the graph. The six strongest lines in the helium spectrum should be ultraviolet (invisible to the eye, 388.8 nm), blue-purple (447.2 nm), skip two weaker peaks, green (501.6 nm), yellow-orange (587.6 nm), red (667.8 nm) and infrared (706.5 nm). Use the cursor tool to accurately determine the pixel position of the top of each of peak and enter it into the corresponding space in the calibration window. Take care to make these assignments correctly – a mistake will cause your subsequent measurements to have errors. The hand-held spectrometer may be helpful in figuring out which peak is which. Then click “Fit Spectrum” to perform the calibration, which should convert the horizontal axis of the graph to a wavelength scale.

L2.4 Click “Done with calibration” to return to the main window. Then click “Acquire single spectrum” to obtain a new helium spectrum with a wavelength scale. If the six largest peaks don’t show the correct wavelengths (to within a few nanometers), then your calibration must be repeated. When finished, print your calibrated graph for your lab notebook, clearly labeling the colors and wavelengths of all six peaks.

Do not close the VI – we plan to use the calibration tool to collect additional spectra.

B. Hydrogen Emission Spectrum

L2.5 Acquire a hydrogen spectrum using a hydrogen Geisler tube. Because the hydrogen tubes have relatively short lifetimes, please use a foot pedal to turn your tube on and off, which avoids leaving the tube on too long.

L2.6 The emission lines of hydrogen are calculated as \( \frac{1}{\lambda} = R \left( \frac{1}{n_f} - \frac{1}{n_i} \right) \), where \( R = 0.010974/\text{nm} \) is the Rydberg constant, and where \( n_i \) and \( n_f \) are the principle quantum numbers of the initial and final energy levels of a given transition. The well-known Balmer-series lines defined by \( n_f = 2 \) include four peaks in the visible spectrum (violet, blue, blue-green and red). Identify the four visible Balmer-series lines with the peaks in your spectrum. The hand-held spectrometer should aid in the identification. If you move the Geisler tube back far enough that the intense red peak doesn't saturate the detector, then the violet peak will be too weak to see. Go ahead and saturate the strongest peaks in order to ensure that the violet peak is visible.

L2.7 Use the cursor tool to measure the wavelengths of all four peaks to the nearest nanometer. Check your results against an online reference. Print your graph for your lab notebook and indicate the colors and wavelengths of each of the four peaks.
L2.8 Use the wavelengths that you measured to calculate $n_i$ (should be an integer) for each Balmer line.

C. Incandescent lamp spectrum.

L2.9 Direct the fiber optic towards an incandescent light bulb (use your desk lamp), collect a spectrum and apply the wavelength calibration. Keep the light bulb sufficiently far from the fiber optic so as to avoid saturating the detector (which cuts the top off of the peak). If the fluorescent room lights are off, one should even be able to collect an incandescent spectrum from across the room. Print the graph for your lab notebook.

L2.10 Compare/contrast the shape of your spectrum to that of an ideal blackbody spectrum. The non-idealities actually lie with the spectrometer rather than the incandescent bulb. Estimate the peak position of your spectrum and use Wien's Displacement Law to estimate the temperature of the filament.

D. Fluorescent lamp spectrum.

L2.11 Direct the fiber optic towards one of the fluorescent lighting panels in the room, collect a spectrum and apply the wavelength calibration. Observe that pointing the fiber optic directly at a fluorescent lamp easily saturates the detector. To avoid saturation, point the fiber optic slightly away from the panel (or even towards the floor or one of the walls). Use the cursor tool to measure the wavelengths of each of the larger and more well-defined peaks in your spectrum. Print a nice-looking unsaturated spectrum for your lab notebook, and record the wavelengths of the major peaks.

L2.12 Refer to http://en.wikipedia.org/wiki/File:Fluorescent_lighting_spectrum_peaks_labelled.png to identify the origin (e.g. solid-state phosphor, elemental gas, etc.) of each of the peaks that you measured in the previous step. Explain how a fluorescent lamp produces white light. What is the physical difference between the light emitted from a ‘cool white’ fluorescent tube and a ‘warm white’ incandescent bulb?

L2.13 Imagine your fluorescent spectrum as a blackbody spectrum by mentally broadening the peaks until they overlap. Estimate the centroid position of the spectrum and use Wien's Displacement Law to determine its effective temperature. Compare this to the published color temperature (4100 K) of an Octron F032/741 bulb.

L2.14 LED – Measure and describe the spectrum of a white LED flashlight bulb. Is this what you expected? Print the spectrum for your notebook and explain the physical origins of its characteristic features.
Lab 3

Interference and Diffraction

In this lab you will quantitatively study diffraction of light and gain experience in making precision optical measurements.

3.1 Spectral Lines

In lab 1, we studied *ray optics* where light is considered to follow straight geometric paths. This works well for the large overall features of light. In contrast, on a smaller scale, we must consider light as a wave, which spreads out after passing through a narrow slit, bends around a corner, or overlaps on itself. In this case, we must treat light using *wave optics*.

The principle of *wave superposition* states that when two light waves pass through the same region of space, the resulting *amplitude* is the sum of the amplitudes of the two individual waves. When the waves add together in phase to produce a larger amplitude, we speak of *constructive interference*. When they add together out of phase to cancel each other out (either partially or completely), we speak of *destructive interference*. When many waves are superimposed to produce a single resultant wave, the interference is called *diffraction*.

It is easiest to observe the wave properties of laser light because it is both *monochromatic* (one color) and *coherent* (all of the wavelets emitted from the source are in phase with one another so as to create well-defined crests and troughs). It is more difficult to see wave interference effects from ordinary light sources (e.g. sunlight, incandescent and fluorescent bulbs, LEDs, etc.), which are generally *polychromatic* (i.e. a mixture of many colors) and *incoherent* (a mixture of many wave phases).

3.2 Double-slit Interference

The simplest manifestation of the wave nature of light is double-slit interference, as depicted in Fig. 3.1. When a coherent light wave impinges on the two slits from the left, the wavelets exiting from each slit are in phase. However, as each wavelet travels a different distance $L_1$ and $L_2$ to a point on a distant screen, their
phases become different. This phase difference is \( \delta = 2\pi (L_1 - L_2)/\lambda \), which leads to a superposed amplitude proportional to \( A \propto \cos \delta \). The intensity of the light (which our eyes detect) varies as the square of the amplitude: \( I = A^2 \propto \cos^2 \delta \). The interference maxima (i.e. constructive interference) occur when the path difference is a whole number of wavelengths:

\[
L_2 - L_1 \equiv d \sin \theta = m\lambda \quad \text{(maximum)} \quad (3.1)
\]

where \( m \) is any integer. The minima (destructive interference) occur in between the maxima at angles that obey

\[
L_2 - L_1 \equiv d \sin \theta = (m + 1/2) \lambda \quad \text{(minimum)} \quad (3.2)
\]

For small angles \( \sin \theta \equiv y/L \), where \( y \) is the vertical position on the screen (see Fig. 3.1) and \( L \) is the distance from the slits to the screen.

### 3.3 Diffraction gratings

When a double-slit arrangement is replaced by a diffraction grating consisting of many equally-spaced slits, Eq. (3.1) still governs the interference maxima, although the peaks become very sharp. A diffraction grating has three key parameters, 1) slit spacing, 2) slit width and 3) slit number. For example, a narrower slit width yields a broader diffraction envelope, within which one can observe many interference peaks, but doesn't change the width or spacing of the peaks. A smaller slit spacing yields a larger separation between two adjacent interference peaks, but doesn't change the width of the peaks or the size of the envelope. A larger number of slits yields sharper interference peaks, but doesn't change the peak spacing or the size of the envelope.

You can browse an introductory physics textbook or the Hyperphysics sections on Interference and Diffraction for more detailed information. Good resolution refers to the ability to distinguish between peaks arising from different wavelengths (when the grating is illuminated with more than one wavelength). A grating has much higher resolution than a double-slit arrangement.

### 3.4 Diffuse Light Source

One can use a grating to measure the wavelength of a monochromatic source, so long as the slit spacing is not wider than the spatial-coherence of the light. This is a piece of cake if you illuminate the grating with a laser; all of the slits get the same phase. However, in research one often desires to characterize the spectrum of a non-laser or diffuse light source. This presents a challenge because the wavefronts from such a source are typically incoherent, meaning the phases of the light across a region of space are jumbled. Without taking special measures, the each grating slit would be somewhat random phase and ruin the grating diffraction pattern.
In this lab, you will study the spectrum from a mercury vapor discharge lamp, which like any diffuse source has poor spatial coherence. We will get around this problem with the setup illustrated in Fig. 3.3. First, the light from the lamp is concentrated onto a single narrow slit using an auxiliary lens. A lens (E-1) placed after the narrow slit is used to collimate the light. This creates nice spatially coherent wavefronts to illuminate the lens.

After the light goes through the grating, we could observe the diffraction pattern on a far-away screen. However, because we lose so much light in our effort to make it spatially coherent, it will be rather hard to see. We can image the pattern that would have appeared on a distant screen to the focus of a second (E-2). This creates a tiny diffraction pattern at the focus of the lens. The angles of the peaks scale with distance. In place of $\sin \theta \equiv y/L$ for a distant screen, we have $\sin \theta \equiv y/f$, where $f$ is the focal length of the lens (E-2). Because the pattern is so small, you will use a microscope to look at it. Tick marks in the microscope will allow you to conveniently measure the inter-maxima spacing and hence the wavelength of the light source.

### 3.5 Equipment

Mercury discharge lamp, optical bench with lens holders, optics set (slits, lenses, mirror, grating, micrometer eyepiece).
Quiz

Q3.1 While type of light source is both monochromatic and highly coherent?
(a) sunlight
(b) neon lamp
(c) red light-emitting diode
(d) blue laser
(e) common fluorescent bulb

Q3.2 Red $\lambda = 635$ nm light that passing through a $d = 0.1$ mm double-slit aperture illuminates a large white screen 75 cm from the aperture. At the center of the resulting interference pattern, compute the angular separation (in radians) and the distance (in cm) between two adjacent peaks.

Q3.3 Qualitatively describe what happens to a grating diffraction pattern when we increase the number of slits while holding the slit width and slit separation constant. Visit the Hyperphysics website and browse the section on *Light and Vision → Diffraction → Multiple Slits*.

Q3.4 In Fig. 3.3, the purpose of the “auxiliary lens” (at left) is to
(a) remove unwanted rays of light that are heading in undesirable directions.
(b) concentrate light from the lamp onto a slit-shaped source object.
(c) correct aberrations in the wave front.
(d) collimate the light (i.e. make the beam parallel and the wave fronts flat).
(e) focus the diffraction pattern onto the plane of the scale.
(f) create a focused image of the “experimental slits”.
(g) transfer an image of the diffraction pattern onto the viewer’s retina.

Q3.5 Consider the two diffraction patterns shown in Figs. 3.4 and 3.5. Which one is a double-slit pattern and which is a single-slit pattern?
Exercises

A. Preliminary: Work together as a class to view the double-slit patterns from two lasers, 473 nm (blue) and 633 nm (red), on a distant white wall, as shown in Fig. 3.1. Use the smallest double-slit spacing available on your slide. Measure the fringe spacing, and use the relation $\frac{\lambda}{d} = \frac{\Delta x}{L}$ to determine the slit separation $d$. Check the slit separation using the microscope eyepiece with adjustable cross hairs (see part 2 of L3.7).

B. Configuration: Set up the equipment as illustrated in Fig. 3.3 without any slits in the slit holder. Make sure that the vertical height of the lamp is positioned within its lamp shield so as to optimize the intensity emerging from the output ports (three ports per lamp). If multiple teams are using different ports of the same lamp, this height cannot be adjusted again once the subsequent alignment steps are underway. Each port has a removable white plastic diffuser that allows you to dim the intensity of the light. When the diffuser is not in place, you should avoid looking directly into the port. After this alignment procedure has been completed, you will probably want to remove the diffuser to maximize the intensity of the interference pattern. Now take the following steps to align the optics. You will save a great deal of time by being very careful.

L3.1 Set the center of all three lenses, the source slit, the micrometer eyepiece, and the slit holder at the same vertical height as your light port to within a few millimeters. Be careful to maintain this vertical alignment throughout the lab. Don't bump the lamp after alignment!!!

L3.2 Orient the optical bench so that it is roughly parallel to the direction of the port that you are using. Roughly set the positions of the optical elements on the optics bench. Place the micrometer eyepiece near the far end of the optical bench. Place the E-2 lens about 20 cm from the eyepiece. (Note that the focal length of lens E-2 is approximately 20 cm). Place the E-1 lens about 15 cm from the first. Place the source slit about 20 cm from the second lens. Place the auxiliary lens (which is probably labeled "M") so that the light from the lamp is nicely focused on the source slit. You will need to remove the diffuser for this.

L3.3 (a) Carefully slide your optical bench from side to side until the beam is centered on the source slit.
(b) By using a sheet of white paper, you should be able to see a bundle of light coming from the source, going through the source slit, passing through each of the lenses, and focusing somewhere near the center of the eyepiece. If this is not the case, try adjusting the orientation of the optical bench. This will mean repeating steps 2 and 3 again afterwards. If necessary, go back to step 1 and be more careful.
(c) You must now reposition lens E-1 so that the source slit is precisely at its focal length. To do this, use a flat mirror to reflect the light back
Exercises

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through the lens to form an image precisely in the plane of the source slit. Use a sheet of paper for a screen.

L3.4 Now cover the light source with the diffuser and look into the eyepiece. Adjust the position of lens E-2 until you see a clear image of the source slit. This image should be near the center of your field of view in the eyepiece. If it is not, adjust the position of the source slit slightly sideways. If you find that this is a large adjustment, you may need to go back to step 3 again. Just to be sure that everything is properly aligned, insert the grating into the slit holder. You should clearly see a multicolored diffraction pattern with a bright white line at the center of the eyepiece.

C. Qualitative observations

L3.5 Insert the green filter so that only the green mercury line is visible at the eyepiece. Then insert the single-slit slide (containing slits of various widths) between the E-1 and E-2 lenses. Which of the single slits does the best job of resolving the higher-order lines, the widest or the narrowest? Explain why.

L3.6 Change filters to isolate the red line and try again. Which color has lines that are easier to resolve, red or green? Explain why.

L3.7 Now insert the double-slit slide (containing slit pairs with various spacings) between the E-1 and E-2 lenses. Use the aluminum aperture to make sure that only one pair is illuminated. Which slit pair has the best resolving power, large or small spacing? Explain why.

L3.8 Finally, insert the grating in place of the double-slits, and compare its resolution to that of the single and double slits.

D. Quantitative wavelength measurements

L3.9 For each of two different mercury lines (i.e. colors) that you select, identify the filter (or filter combination) that best isolates it. Then measure the wavelength of each line twice, once using the best double-slit pattern and once using the grating. It will be helpful to note the following:

1. For small angles (expressed in radians) \( \theta \approx \frac{y}{f} \) where \( y \) is the displacement of a given line relative to the center of the eyepiece scale (one full turn is 0.01 inch), and \( f \) is the 20 cm focal length of the E2 lens.

2. Slit-spacings can be accurately measured using the low-power microscope in the lab which has a ruled eyepiece. One full turn on the microscope dial is 0.002 inch.

3. A more accurate measurement of the spacing can be obtained by measuring the width of the entire pattern and then dividing the number of maxima in the pattern.
Lab 4

Radioactivity: Curve Fitting

In this lab you will study radioactive decay as a context for exploring the concepts, tools, and techniques of data analysis and curve fitting.

IMPORTANT: You must take care to save your data from this lab; it will be used again in the next lab where you will further analyze it.

4.1 Radioactivity

There are a growing number of useful applications for radioactive sources. Radioactivity is the name given to an atomic or nuclear process in which the nucleus spontaneously decays by throwing out one or more of its constituent parts. The nucleus often transforms into a different element of the periodic table in this process. In radioactive decay, a nucleus emits one or more of the following high-energy particles: beta ray (a fast moving electron), alpha ray (a fast moving $^4$He nucleus, consisting of two protons and two neutrons), neutron, or gamma ray (a high-energy photon).

In contrast to microwave or visible light radiation, radioactive sources produce ionizing radiation, meaning that the emitted particles have enough energy to free electrons from atoms in collisions – hence the possibility of creating chemical changes that can harm biological tissue. Because the radioactive sources that we use in this lab ($^{137}$Cs) are sealed (to prevent spreading the material around), and because they are weak sources to begin with, the radiation levels that you will be exposed to are very low – comparable to the natural background radiation received from a few days of hiking in the mountains. Nevertheless, it is a good idea not to hold the radioactive sources for long periods of time, carry them in your pocket, or far worst – eat them. Common sense and good hygiene are essential to safety in any laboratory.

You will use a standard Geiger counter, which is an inert gas-filled tube that briefly conducts electricity when an energetic particle passes through and ionizes some of the gas. While Geiger counters are most commonly used for alpha and beta ray detection, they can also be used for gamma detection. The cesium 137
source that you will use emits betas and gammas.

4.2 Exponential Radioactive Decay over Time

The spontaneous decay of an individual atomic nucleus is not predictable but is a chance process described statistically – there is a fixed but small probability that a given nucleus will decay in any given time interval. Normally, we measure radioactivity from a large collection of atoms. We count clicks or events on a Geiger counter, a device that detects and registers individual radiation particles as they pass through it. We can, for example, record the decay rate \( R \) in units of events per second. If you place a Geiger counter some distance from a radioactive source, a fixed fraction of the particles emitted from the sample will pass through the counter and be measured. Since there are an enormous number of atoms (e.g. \( > 10^{18} \)) in any macroscopic sample, it is typical to record many decay events each second.

Consider a sample containing \( N \) nuclei. The basic concept of radioactive decay is expressed by the equation

\[
\frac{\Delta N}{\Delta t} = -\lambda N \tag{4.1}
\]

where \( \Delta N \) is the number of nuclei decaying within a time interval \( \Delta t \) and the decay coefficient \( \lambda \) is the probability per time that a given nucleus will decay. We see that the decay rate \( R \equiv -\Delta N/\Delta t \) (or \( -dN/dt \)) is proportional to the number of atoms in the sample \( N \). The solution to Eq. (4.1) is the well-known exponential decay formula:

\[
N = N_0 e^{-\lambda t} \quad \text{and} \quad R = R_0 e^{-\lambda t} \quad \text{where} \quad R_0 \equiv \lambda N_0 \tag{4.2}
\]

\( N_0 \) is the initial number of nuclei at the time arbitrarily defined as \( t = 0 \). The decay rate falls off exponentially with time as shown in Fig. 4.1.

After a sufficient time, called the half-life \( (T_{1/2}) \) of the radioactive nucleus, the number \( N \) of not-yet-decayed nuclei that remain will be equal to one-half of the initial number \( N_0 \). If we start with a decay rate of \( R_0 \), the rate after after a period of three half-lives reduces to \( R_0 / 2^3 \). Because \(^{137}\text{Cs}\) has a long 35-year half-life, the count-rates that you observe will not vary appreciably during the course of the laboratory period.

**Example 4.1**

Determine the half-life \( T_{1/2} \) in terms of the decay coefficient \( \lambda \).

**Solution:** At time \( t = T_{1/2} \), the remaining number of not-yet-decayed nuclei is \( N = N_0 / 2 \). Substituting this into Eq. (4.2) gives

\[
\frac{N_0}{2} = N_0 e^{-\lambda T_{1/2}} \quad \Rightarrow \quad e^{-\lambda T_{1/2}} = 2 \quad \Rightarrow \quad \lambda T_{1/2} = \ln 2
\]
4.3 Radiation Absorption

The electrical charge of alphas and betas cause them to interact strongly with materials. Alpha particles are readily absorbed by the air. Beta particles will penetrate a few inches of air, but are easily stopped by a thin sheet of metal. While the details of this absorption are quite complex, suffice it to say that most alpha and beta particles are absorbed or scattered after traveling a rather short distance in a material. Gamma rays, on the other hand, interact only weakly with matter (typically via Compton scattering), and therefore are much more penetrating. You will study this in the present experiment.

As with spontaneous decay of radioactive nuclei, the absorption of gamma radiation is a random statistical process. Individual gamma-ray photons may or may not interact with individual atoms in a material. These gamma rays are thus lost (scattered and eventually absorbed) from the transmitted beam of gamma rays that would have otherwise entered the counter. If $N$ gamma rays would have passed from the source into the counter without the absorbing material present, the number $\Delta N$ lost from the beam by its passing through a thickness $\Delta x$ of the material is given by

$$\frac{\Delta N}{\Delta x} = \mu N \quad (4.4)$$

where $\mu$, called the absorption coefficient, is the probability of scattering of each gamma ray per length of material through which it passes. The absorption coefficient has units of inverse distance (e.g. 1/mm). The analogy with Eq. (4.1) is obvious, and the decay in the count rate of gammas in the transmitted beam as a function of distance in the material also decays exponentially, as shown in Fig. 4.2.

$$R = R_0 e^{-\mu x} \quad (4.5)$$

The inverse of the absorption coefficient, $1/\mu$, provides a measure of how deeply the radiation penetrates the absorbing material. In analogy with the half life, the absorption half-length $x_{1/2}$ is a measure of the penetration depth for which absorption reduces the radiation count rate to half its original value. The absorption coefficient and the half-length are related according to (compare with example 4.1)

$$x_{1/2} = \frac{\ln 2}{\mu} \quad (4.6)$$

4.4 Fitting Mathematical Expressions to Experimental Data

One often desires to fit a mathematical function to an experimental dataset. Typically, the function has one or more undetermined parameters that can be
adjusted to obtain a *best fit* to the data. The function of choice may convey a relationship that is based on a physical model, or it may be purely *empirical* (i.e. chosen for no reason except that it seems to work). Before attempting to fit a function $y(x)$ to a set of $N$ points on a graph: $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$, always plot the raw data first to get a visual ‘feel’ for the trend, and then make an intelligent guess regarding a suitable functional form. In many cases, theoretical arguments will suggest a function that is likely to work. For example, if the trend appears to be linear, one could employ a function of the form $y = ax + b$, where $a$ and $b$ are called the *fitting parameters*.

Once you have selected a function, the next step is to determine the values of its undetermined parameters that will best fit your data. If the function $y(x)$ is a perfect fit (unlikely), then $y_i = y(x_i)$ for every data point. More typically, there will be discrepancies between the observed $y_i$ and the calculated $y(x_i)$. In this case, a good measure of the overall discrepancy can be computed as

$$E(a, b, \ldots) = \sum_{i=1}^{N} (y_i - y(x_i))^2$$ (4.7)

$E$ is known as the *least squares error* or *badness-of-fit*, and depends on any fitting parameters in the function $y(x)$. In practice, the *best fit* is obtained by employing a computer algorithm to vary the fitting parameters until the badness-of-fit is minimized, a process commonly known as *least squares minimization*.

As an example, when the line $y = a + bx$ is used fit to the data, then $E(a, b) = \sum (y_i - (ax_i + b))^2$. In this case, an analytic solution exists for the values of $a$ and $b$ that will minimize $E$ and provide the best fit to the data $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$. Most calculators and spreadsheet programs have a linear regression function that relies on this solution (not given here). More complicated functions that appear frequently include power laws and exponentials, both of which can be manipulated into linear form so as to take advantage of straight-line simplicity. When *linearization* is possible, seeing the modified data fall along a straight line provides a nice visual verification of a good fit.

### Example 4.2

Manipulate the power function $y = b + ax^m$, where $a$, $b$, and $m$ are fitting parameters, so that a linear fit may be used.

**Solution:** One can apply the natural logarithm to both sides of the above expression to obtain

$$\ln(y - b) = \ln(a) + m \ln(x)$$

Defining $u = \ln(y - b)$ and $v = \ln(x)$ produces linear function

$$u(v) = \ln(a) + mv$$

which has a $u$ intercept of $\ln(a)$ and a slope of $m$. By calculating $u_i = \ln(y_i - b)$ and $v_i = \ln(x_i)$ at each data point, we can fit $u(v)$ to the $(u_i, v_i)$ data rather than directly fitting $y(x)$ to the more complicated $(y_i, x_i)$ data.
Example 4.3

Manipulate the exponential function \( y = ae^{mx} \), where \( a \) and \( m \) are fitting parameters, so that a linear fit may be used.

**Solution:** As in example 4.2, the expression can be put into linear form by taking the natural logarithm of both sides:

\[
\ln(y) = \ln(b) + mx
\]

By defining \( u = \ln(y) \), we arrive at a new linear expression of the form

\[
u(x) = \ln(b) + mx
\]

with a \( u \)-intercept of \( \ln(b) \) and a slope of \( m \). After calculating \( u_i = \ln(y_i) \) for each data point, \( u(v) \) can be fitted to the \((u_i, x_i)\) data.

With functions that cannot be manipulated into linear form, one is stuck fitting the data to some sort of curve. Fitting more complicated functions typically requires a **nonlinear least-squares** (NLSQ) fitting algorithm. There are many general algorithms for NLSQ fitting, which tend to be classified according to their method of exploring the available **fitting-parameter space**. A difficulty with NLSQ is that there is no guarantee that the ‘best’ fit will be found; the algorithm may get caught in a local minimum of \( E \), providing a ‘good’ but not ‘best’ solution.

### 4.5 Equipment

Geiger counter and sample holder, weak sealed radioactive Cesium 137 sources, stack of aluminum and lead absorber plates.
Quiz

Q4.1  The $^{137}$Cs source emits both gamma rays (energetic photons) and beta rays (energetic electrons). These two types of radiation penetrate through a material barrier (choose one)
(a) with the same rate of absorption.
(b) with different rates of absorption.

Q4.2  The count rates that you will read with a Geiger counter will (choose one)
(a) include background radiation.
(b) not include background radiation.

Q4.3  Why does taking the natural logarithm of both sides of Eqs. (4.2) or (4.5) allow us to fit a straight line to exponentially decaying data?

Q4.4  The count rate of gamma rays after passing through a 3 mm sheet of metal with absorption coefficient $\mu = 0.7 \text{cm}^{-1}$ is 1500 counts/sec.
(a) Determine the gamma-ray half-thickness of the metal.
(b) Determine the initial gamma-ray count rate before the beam hits the metal plate.
Exercises

A. Get familiar with your equipment.

L4.1 To become familiar with the counting apparatus shown in Fig. 4.3, first place your $^{137}$Cs source with the top facing up under the counting chamber without absorbers. Turn the counter on and slowly increase the high voltage up to 900 volts. Do not let the voltage exceed 900 volts—the expensive Geiger tube can be easily destroyed by excessive voltage. Practice counting for an interval of 60 seconds.

L4.2 You will see a small background count rate at all times due to cosmic rays and radioactive materials that occur naturally in the walls of the building. To measure this background, remove the cesium source and all other sources of radiation from the vicinity of the counter (at least two meters away) and observe the counts accumulated during a period of several minutes. Divide the number of counts $N_B$ by the time interval $T_B$ in seconds to get the background count rate $R_B$, and record it in your lab notebook.

L4.3 Using your apparatus as shown in Fig. 4.3, measure the 15-second count rates using several different thicknesses of aluminum. Try a very thin sheet, a medium-thickness sheet, a thick slab and several thick slabs. You will observe that the rate initially falls off quickly with increasing thickness, but falls off more slowly as additional plates are added. Explain that weakly penetrating beta rays contribute to the detected signal at small absorber thicknesses, while only gamma rays penetrate at larger thicknesses.

B: Measure absorption half-length in lead for gamma rays emitted from $^{137}$Cs.

L4.4 Carefully vary the lead absorber thickness from zero up to about 3 cm using the smallest available thickness increment (i.e. the relatively thin sheet of lead). For each thickness, count the number of gamma rays detected within a 60-second interval, and record ALL of your raw data (e.g. thickness in mm, number of counts, and counting interval in seconds) in your lab notebook. The total number of data points should be about 20.

L4.5 Open an Excel spreadsheet called dataanalys.xls. Enter your data into this spreadsheet as three columns named “thickness(t)”, “interval(T)”, and “counts(N)”. Add another column containing the “background($R_B$)” count rate, which is the same for each data point. Next, add a column that computes the total count rate $R = N/T$ in counts/second, and another column that computes the natural log of the background-subtracted count rate, $\ln(R - R_B)$. Print this table for your lab notebook.
L4.6 Use Excel to generate an $XY$ scatter plot of $R$ as a function of thickness (in mm) for your lab notebook. Do not use guide lines. Display the data points as open circles. Make sure that all axes are properly labeled including the correct units and print the plot for your lab notebook.

L4.7 Make a similar plot for the value of $\ln(R - R_B)$ as a function of thickness. You should observe that the $\ln(R - R_B)$ graph has a well-defined linear slope. Use Excel's linear-regression feature to measure this slope (in $1/mm$ units) – simply right click any data point in the plot, select add trendline, and select the option that displays the fitted equation. Add the plot to your lab notebook.

L4.8 (a) Mathematically demonstrate (in your notebook) that the slope of your $\ln(R - R_B)$ curve should be equal to $\mu$. (b) Use the slope value that you calculated above to estimate the absorption half-length for gamma-rays in lead. (c) Also use the fitted intercept value to estimate the initial gamma-ray count rate in the absence of any lead.

Be certain to save your Excel spreadsheet on the computer; you will need it for the next lab.

C. Perform a non-linear least-squares fit.

L4.9 Create a two-column dataset in Logger Pro. Name the columns “Thickness” and “Rate” and provide the correct units for each. Copy and paste the data from the corresponding columns in your Excel worksheet. Figure out how to use the column options to display the data as open circles.

L4.10 (a) Use Logger Pro to fit a non-linear model of the form $R(t) = a e^{-\ln(2)t/c} + b$ to your data. If your model was entered correctly, your fit should converge easily.

(b) Print the associated plot and fitting results (i.e. values and units) in your lab notebook.

(c) Explain what physical quantity each fitting parameter represents. You may want to emphasize this in your conclusion as well.

(d) How do the values for background level, initial rate and half-length obtained from this non-linear fit (using Logger Pro) compare to those obtained from a linear fit to linearized data (using Excel)?

Be certain to save your Logger Pro file on the computer; you will need it for the next lab.
Lab 5

Radioactivity: Experimental Uncertainty

In this lab you will learn about statistical distributions of random processes such as radioactive counts. You will also further analyze the gamma-ray absorption data that you obtained in your last lab. You will learn about statistical error, error propagation, non-linear curve fitting, and fitting uncertainties.

5.1 Experimental Error: Accuracy vs Precision

It is very common to hear the word *error* used loosely in reference to experiments. By this, we usually mean *uncertainty* in a result (as opposed to a blunder). When in fact, there are different types of error that should be carefully distinguished. Whenever using the word error in your lab reports, you should be careful to explicitly indicate the type and the source of the error that you are referring to.

Accuracy Limitations

The *accuracy* of a measurement refers roughly to the *maximum difference* between the experimentally measured value and the expected or idealized value (sometimes unknown). We sometimes talk of massless ropes, point masses, perfect insulators, frictionless surfaces, etc. However, real experiments often require one to take into account systematic non-idealities such as friction. These *systematic errors* limit the accuracy of an experiment.

Does a given systematic error indicate a flaw in the theory or in the experiment? This is really a matter of perspective. If one fails to use a sufficiently complex theory to account for all of the variables, then we might say that the theory is flawed. On the other hand, if one intends to measure the influence of a single variable, but fails to eliminate or minimize the influence of other variables, we could say that the experiment is flawed. In any case, we must carefully isolate systematic errors and either include them in the theoretical model or else eliminate them from the experimental data. Often, this requires starting over.
Precision Limitations

Even with a theory that accurately represents your experiment, there will always be a limit to the precision of your measurements. Precision refers to the smallest increment in the experimentally measured value that can be reproducibly controlled. In principle, if all systematic errors have been eliminated, the measurement accuracy becomes equal to the measurement precision.

Two obvious limitations to experimental precision are instrument resolution and statistical error. A ruler with only millimeter resolution, for example, can’t be expected to provide 1 micron of precision, and a digital bathroom scale with a 0.1 kg resolution can’t be expected to provide 1 gram of precision. Moreover, if you repeat the same experiment many times using a sufficiently sensitive technique, you will find that the outcome varies from one measurement to the next due to statistical randomness, producing a distribution of measured values. If, for example, you use a hand-held millisecond stop-watch to time the fall of a mass from a 1-meter height, and repeat the measurement several times, you will almost certainly get different results due to random errors in your motion perception and reaction speed. In this lab, you will quantify practical precision limits due to instrument resolution and statistical error.

5.2 Histograms and Probability Distributions

When you make a reasonably large number of measurements, a histogram-type graph of the results can help you to visualize the amount of variation in the data and check for abnormalities in the overall distribution. The histogram (see Fig. 5.1) is a bar graph obtained by grouping together all measurements that fall within predetermined intervals of the random variable \(x\). You must judiciously select the size of the interval \(\Delta x\), such that a suitably large number of measurements get grouped into each interval. If you slice the intervals too finely, and only wind up with 0 or 1 counts in each interval, the shape of the distribution is difficult to discern. As \(N\) grows larger, the size of the intervals can be reduced while retaining a modest number of counts in each interval, so that the histogram gradually takes on the true shape of distribution.

As \(N\) becomes very large, the interval width can be made very narrow and the histogram curve approaches a smooth curve, as illustrated in Fig. 5.2. Such a curve is called a probability distribution function (PDF). A probability distribution is typically normalized, meaning that the amplitude of the function is adjusted such that \(\int f(x) \, dx = 1\). Integrating over a sub region, such as the shaded region in Fig. 5.4, then yields the probability that a measurement will give an outcome in that range.

The Gaussian Distribution

Many physical processes exhibit intrinsically random variations that obey a probability distribution function commonly referred to as the Gaussian or normal
distribution, which has the mathematical form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$  \hspace{1cm} (5.1)$$

Fig. 5.4 shows this distribution.

**Counting Statistics: The Poisson Distribution**

Suppose that we count the number of random events that occur during a finite time interval. The probability of obtaining a specific count \((n)\) obeys a poisson distribution, which has the form

$$f(n) = \frac{N^n e^{-N}}{n!}$$  \hspace{1cm} (5.2)$$

where \(N\) is the expected mean number of counts. In contrast to the Gaussian distribution, the Poisson distribution is discrete rather than continuous, meaning that \(n\) must be an integer value. If large numbers of counts are involved, the Poisson distribution takes an approximately Gaussian shape.

**Example 5.1**

Suppose that the number of cars that pass through a certain intersection interval is completely random. Suppose further that after checking many 10-minute intervals you determine that the average number of cars passing in that amount of time is \(N = 18.7\). What is the probability that during a given 10-minute interval, 12 cars will pass?

**Solution:** Since the number of cars passing is random, the outcome is subject to Poisson statistics Eq. (5.2). The probability that \(n = 12\) cars will pass during a given 10-minute interval is

$$f(12) = 18.7^{12} e^{-18.7} / 12! = 0.0289 \approx 3\%$$

### 5.3 Statistical Measures: Mean and Standard Deviation

**Mean Values**

Suppose that we make \(N\) measurements of the distance between two points. These measurements, which could be made by one person or by different people with different measuring tapes, may produce a range of results. If we assume that the errors are random with equal likelihood of being positive or negative, then the average of all \(N\) measured values would be the best estimate of the true distance. Of course, any systematic errors (for example, caused by a flawed tape measure) might add an offset (also called a bias) to the statistical errors, which would need to be separately considered. While a systematic error would impact
5.3 Statistical Measures: Mean and Standard Deviation

accuracy, it would not necessarily impact precision, which has to do with the statistical repeatability of the measurement. For present purposes, we assume that any errors are purely statistical, fluctuating around the true value.

Designate the $i^{th}$ measured distance as $x_i$. The mean (or average) of $N$ measured values is computed as

$$\bar{x} = \frac{1}{N} (x_1 + x_2 + x_3 + \cdots + x_N) = \frac{1}{N} \sum x_i$$  \hspace{1cm} (5.3)

Alternatively, consider that you accurately measure the diameters of $N$ individual hairs from your head, once again arriving at a distribution of results, not limited by precision but because each hair has a slightly different diameter. You could designate each individual measurement as $x_i$, and calculate the mean hair diameter using Eq. (5.3). In this case, there is no such thing as the ‘true’ diameter. $\bar{x}$ simply represents the average of the distribution.

**Standard Deviation**

Suppose we perform $N$ individual measurements of nominally the same thing and that these measurements fluctuate about a mean $\bar{x}$. We can compute the individual deviations of the data points from the mean as follows: $d_i = x_i - \bar{x}$. Since some of the deviations are positive and others negative, the average $\bar{d} = \frac{1}{N} \sum d_i = \frac{1}{N} \sum(x_i - \bar{x})/N = (\bar{x} - \bar{x})/N$ is precisely zero. Alternatively, one can take the average of the absolute value of deviations, which is called the probable error: $\text{P.E.} = \frac{1}{N} \sum |x_i - \bar{x}|$.

More commonly, however, one computes the root mean square of the deviations:

$$\sigma \equiv \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}} = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \cdots}{N}}$$  \hspace{1cm} (5.4)

$\sigma$ is called the standard deviation of the distribution. The square of the standard deviation (i.e. $\sigma^2$) is called the variance. This formula could be used in the previously mentioned example of measuring hair widths. The standard deviation gives a quantitative measure of the spread or scatter of the measurements around the average value $\bar{x}$. For a large number of measurements, $\sigma$ is associated with the effective width of the associated probability distribution.

For Eq. (5.4) to make sense, one should take enough measurements to get a good idea of the scatter. One often replaces it with an improved formula, which divides by $N - 1$ instead of $N$:

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N - 1}}$$  \hspace{1cm} \text{(improved formula)}  \hspace{1cm} (5.5)

For large values of $N$, this makes essentially no difference. But for small (inappropriate) values of $N$, Eq. (5.5) exaggerates the error to a more realistic value, given the lack of evidence for the range of variation.

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5.4 Propagation of Measurement Errors to Derived Quantities

Statistical Properties of the Gaussian Distribution

These properties are typically proved in formal statistics courses, but are quoted here for convenience. These rules break down if the distribution is not a truly Gaussian.

1. For a large number of measurements, 68.3% of measured values $x_i$ will lie within a standard deviation of the mean, and 31.7% will lie outside this range.
2. The probable error $P.E.$ has the very simple interpretation that 50% of the measured values lie within the range $\bar{x} \pm P.E.$ and 50% outside of that range. The probable error is $P.E. = 0.674\sigma$. 
3. Similarly, there is a 95.5% that a given measurement will lie within the range $\bar{x} \pm 2\sigma$ and a 99.7% chance that it will lie within the range $\bar{x} \pm 3\sigma$.

Example 5.2

Relate the parameter $\sigma$ of the Gaussian distribution in Eq. (5.1) to the full width at half maximum $x_{FWHM}$ for the curve.

Solution: By definition, when $x = \bar{x} \pm x_{FWHM}/2$, the Gaussian distribution $f(x)$ should reduce to half its peak value:

$$f(\bar{x} \pm x_{FWHM}/2) = f(\bar{x})/2$$

That is

$$e^{-\frac{(x_{FWHM}/2)^2}{2\sigma^2}} = \frac{1}{2} \Rightarrow -\frac{(x_{FWHM})^2}{8\sigma^2} = \ln\left(\frac{1}{2}\right) \Rightarrow x_{FWHM} = \sigma\sqrt{8\ln2} \approx 2.35\sigma$$

Statistical Properties of the Poisson Distribution

The Poisson distribution has the interesting property that its variance is equal to its mean. Thus for a mean value of $N$ counts, the standard deviation is $\sigma = \sqrt{N}$. If the average number of radioactive decays detected during a one-minute interval is $N = 100$, the standard deviation is 10, meaning the majority of repeated measurements will fall in the range $100 \pm 10$.

5.4 Propagation of Measurement Errors to Derived Quantities

One often needs to propagate the statistical uncertainties of experimentally measured parameters to other quantities derived from those parameters. Suppose that we determine the speed of an object as the ratio of a measured distance and a measured time interval. In that case, both time and distance are raw data...
variables, while speed is a derived quantity. To determine the error in the speed, we must first estimate the errors in both the time and the distance measurements, and then propagate those errors to the derived speed. For this reason, you should always estimate and record the errors associated with each raw data variable measured. In this lab, you will need to provide meaningful estimates of errors in derived quantities based on the errors in the constituent raw data.

If \( f \) is a derived function of measured variables \( a, b \) and \( c \), which have measurement uncertainties \( \Delta a, \Delta b \) and \( \Delta c \), respectively, the corresponding uncertainty \( \Delta f \) is approximated using the Gaussian error-propagation formula:

\[
(\Delta f)^2 = \left( \frac{\partial f}{\partial a} \right)^2 (\Delta a)^2 + \left( \frac{\partial f}{\partial b} \right)^2 (\Delta b)^2 + \left( \frac{\partial f}{\partial c} \right)^2 (\Delta c)^2 \tag{5.6}
\]

Observe that the overall uncertainty in \( f \) depends not only on the individual uncertainties in \( a, b \) and \( c \), but also on how sensitive \( f \) is to those variables, as dictated by the partial derivatives. A partial derivative (denoted with \( \partial \)) is like an ordinary derivative except it treats all parameters as constants except for the variable being differentiated. Because uncertainties sum nicely after squaring each one, we say that Gaussian errors add in quadrature. Physicists often assume that all relevant variables, including the derived quantity, have Gaussian distributions, even when the actual statistics are not known. In other words, for better or for worse, physicists use Eq. (5.6) a lot.

The Gaussian error propagation formula in Eq. (5.6) is a general formula that can be applied to any non-pathological function. In the following two special cases (additive and multiplicative errors), it can be reduced to an even simpler form.

1. If \( f = \pm a \pm b \pm c \), then

\[
(\Delta f)^2 = (\Delta a)^2 + (\Delta b)^2 + (\Delta c)^2 \tag{5.7}
\]

2. If \( f = a^j b^k c^\ell \), where \( j, k, \ell \neq 0 \) (e.g. \( abc \) or \( a/bc \), etc.), then

\[
\left( \frac{\Delta f}{f} \right)^2 = j^2 \left( \frac{\Delta a}{a} \right)^2 + k^2 \left( \frac{\Delta b}{b} \right)^2 + \ell^2 \left( \frac{\Delta c}{c} \right)^2 \tag{5.8}
\]

The quantity \( \Delta f / f \) is termed the fractional or relative error in \( f \), while the quantity \( (\Delta f / f) \times 100 \) is termed the percent error in \( f \).

**Example 5.3**

Consider the calculated density \( \rho \) of a circular metal cylinder:

\[
\rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{\pi R^2 L}
\]

With a Vernier caliper and an electronic scale, suppose you find that \( M = 185.2 \) g, \( L = 6.23 \) cm and \( R = 1.05 \) cm, from which we calculate the density to be \( \rho = 8.58 \text{g/cm}^3 \). Based on the precision of the measuring instruments, you deduce that
associated errors are $\Delta M = 0.1 \text{ g}$ and $\Delta L = \Delta R = 0.01 \text{ cm}$. Compute the uncertainty in $\rho$.

**Solution:** The partial derivatives needed for applying Eq. (5.6) are

\[
\frac{\partial \rho}{\partial M} = \frac{1}{\pi R^2 L} = \frac{\rho}{M} \quad \frac{\partial \rho}{\partial L} = \frac{-M}{\pi R^2 L^2} = -\frac{\rho}{L} \quad \frac{\partial \rho}{\partial R} = \frac{-2M}{\pi R^3 L} = -\frac{2\rho}{R}
\]

Alternatively, since the density is defined as a product of measured variables, we can employ Eq. (5.8), from which we get

\[
\left(\frac{\Delta \rho}{\rho}\right)^2 = \left(\frac{\Delta M}{M}\right)^2 + (-1)^2 \left(\frac{\Delta L}{L}\right)^2 + (2)^2 \left(\frac{\Delta R}{R}\right)^2
\]

The uncertainty in the density $\rho$ then works out to be

\[
\Delta \rho = \rho \sqrt{\left(\frac{\Delta M}{M}\right)^2 + \left(\frac{\Delta L}{L}\right)^2 + 4 \left(\frac{\Delta R}{R}\right)^2}
\]

\[
= \left(8.58 \ \frac{\text{g}}{\text{cm}^3}\right) \sqrt{\left(\frac{0.1}{185.2}\right)^2 + \left(\frac{0.01}{6.23}\right)^2 + 4 \left(\frac{0.01}{1.05}\right)^2}
\]

\[
= 0.16 \ \frac{\text{g}}{\text{cm}^3}
\]

Based on the above information, we can say that the density of the cylinder is $\rho = (8.58 \pm 0.16) \text{g/cm}^3$. By looking at the above expressions, which quantity would you like to measure more accurately and why?

### 5.5 Quoting Experimental Uncertainties

In reporting experimental results, data are usually listed with a numerical value for the uncertainty. For example, the best value of the mass of an electron currently is $(5.48579902 \pm 0.00000013) \times 10^{-4} \text{ atomic mass units}$. Since uncertainty can be defined in many different ways, how do we know what number to assign? For the purposes of most analyses, and in particular for this class, the rules for assigning uncertainty are as follows:

1. For **statistical uncertainty**, use one standard deviation (unless otherwise specified).
2. For **instrumental uncertainty**, estimate the uncertainty as well as you can. Although this uncertainty isn't strictly Gaussian, treat it as if it were Gaussian in subsequent error analysis.
3. For known **systematic errors** (not really an uncertainty), either the data or the theory should be appropriately corrected in an clearly-stated fashion.

### 5.6 Equipment

Geiger counter and sample holder, weak sealed radioactive Cesium 137 sources, stack of aluminum and lead absorber plates, stopwatch (or watch with a timer).
Quiz

Q5.1 Measurements of a low-intensity long-life radioactive sample with an average rate of 5 counts/sec are accumulated during a 10-second interval. This measurement is then repeated 5000 times to create a histogram. Based on what you have read about the statistics of counting random events during a finite time period, which of the histograms shown in Figs. 5.5–5.8 could represent the results of this experiment?

Q5.2 Define total time $T = T_1 + T_2$ as the calculated sum of two measured times, $T_1$ and $T_2$. Use Gaussian error propagation to compute $\sigma_T$ in terms of $T_1$, $\sigma_{T_1}$, $T_2$ and $\sigma_{T_2}$.

Q5.3 Define velocity $v = D/T$ as the calculated ratio of measured distance $D$ over measured time $T$. Use Gaussian error propagation to compute $\sigma_v$ in terms of $D$, $\sigma_D$, $T$ and $\sigma_T$.

Q5.4 Define velocity $v = D/(T_1 + T_2)$ as the calculated ratio of measured distance $D$ over the sum of measured times $T_1$ and $T_2$. Use Gaussian error propagation to compute $\sigma_v$ in terms of $D$, $\sigma_D$, $T_1$, $\sigma_{T_1}$, $T_2$, and $\sigma_{T_2}$.

Q5.5 Suppose that you separately measure the frequency ($f = 4.74 \pm 0.02 \times 10^{14}$ Hz) and wavelength ($\lambda = 633.0 \pm 0.5 \times 10^{-9}$ m) of a red laser. Recall that the speed of light is $c = \lambda f$. Assuming Gaussian errors, what is the numerical uncertainty in the speed (in m/s) calculated based on this information?
Exercises

A. Experimentally sample and analyze a statistical distribution.

L5.1 Employ the number of aluminum absorbers needed to reduce the detected $^{137}\text{Cs}$ count rate to about 1000 counts in 10 seconds. Using a stopwatch, make 100 separate counting measurements in 10-second time intervals, and record the number of counts for each measurement. This should take you less than half an hour to complete. Do not compute a rate – simply record the number of counts that you actually measured! You will need to work together as a team with a stop watch in order to keep the time-interval errors low. Record the data (just the number of counts) in a new Excel spreadsheet called “poisson.xls”.

If the Data Analysis add-in does not appear in Excel’s Data menu (far right-hand side), then you need to install it: (1) Select Options in the File menu. (2) Select Add-Ins in the popup window. (3) Choose Analysis ToolPak and click Go. (4) Check the box for Analysis TookPak and click OK. Now the add-in should appear in the Data menu.

L5.2 Use Excel’s Data Analysis: Histogram tool to generate a histogram of your data. Use a formula like 800+20*(ROW()) to fill a separate column of your spreadsheet with 11 evenly-spaced bin values that span the range of your dataset. Edit the axes of your histogram, delete the legend, and print the graph for your lab notebook.

L5.3 Perform a basic statistical analysis of your data using the Data Analysis: Descriptive Statistics tool in Excel. Be sure to check the Summary Statistics and Confidence Level boxes. Copy the resulting analysis into your lab notebook.

L5.4 Ideally, you should obtain a Poisson distribution, which means that your variance should be approximately equal to your mean. Check this in your statistical analysis and comment on your observation. If other sources of error have contributed to your variance (e.g. time interval measurement errors), the standard deviation may be larger.

L5.5 For a mean value that is much greater than one, a Poisson distribution is approximately Gaussian in shape. Imagine that your distribution is Gaussian and use the numerical value of the standard deviation to estimate the FWHM (full width at half maximum) of the distribution. (See example 5.2.) Does your result agree well with the FWHM of the histogram that you graphed?

B. Estimate raw data errors, and apply Gaussian error propagation.

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L5.6 Open the dataanalysis.xls spreadsheet that you created previously and add two new columns, “sigmaT” and “sigmaN”, which contain estimates of the errors in your raw data. For counts, use that fact that $\sigma_N = \sqrt{N}$ for a Poisson distribution. Use common sense to estimate the errors in the thickness and in the counting interval.

L5.7 Because the count rate $R \equiv N/T$ is computed from two raw-data variables, $N$ and $T$, you will need to propagate the raw-data errors, $\sigma_N$ and $\sigma_T$, into an estimated error $\sigma_R$. (See example 5.3 and question Q5.3, where $\sigma_N$ is similar to $\Delta N$, etc.) Place this computation into yet another column of your spreadsheet. Estimating errors in quantities that you didn't measure directly is tricky. Consult your TA as needed. Print a copy of the whole table (without plots) into your lab notebook.

L5.8 In your own words, explain the difference between error estimates for raw data and propagated error estimates for calculated quantities.

C. Perform a non-linear least-squares fit with error propagation.

L5.9 Open the Logger Pro file that you created previously and add a column (label it “sigmaR”) for the uncertainty in the count rate. Copy and paste the uncertainty data from the corresponding column in Excel. Regenerate the graph and figure out how to use the column options to display the data as open circles with error bars.

L5.10 Redo the previous fit using the same model, $R(t) = ae^{-\ln(2)x/c} + b$. If the error bars are properly calculated, the fitted curve will typically stay within the error bars of most of the individual data points. If your error bars are too large (data passes nicely through the center of each bar) or too small (the curve misses more than a few bars), go back to Excel, find the problem and fix it. When the fit and the error bars look good (show your TA), plot the graph for your lab notebook. Be sure to include the values of the fitting parameters and their uncertainties!

L5.11 Explain what physical quantity is represented by each of the fitting parameters. Among other things, the half length and its uncertainty should be prominently displayed and interpreted in your report. Calculate the relative error (as a percentage) for both the half length and the background. Do these values seem reasonable?

L5.12 In what sense is this half-length value an improvement over the half-length value obtained in the previous lab? It is important that you understand that without an uncertainty estimate, any experimentally-determined value is essentially meaningless. Your conclusions should convey this understanding. Note that while Excel has basic curve-fitting capabilities, it cannot utilize the error estimates of the individual data values or estimate the uncertainty in a fitting parameter – it’s more of a toy than a serious analytical tool.
Lab 6

AC-Circuit Theory and Complex Numbers

In preparation for future labs, we provide here some basics about AC circuits. (AC stands for alternating current.) We also explain how exponentials of imaginary numbers are equivalent to trigonometric functions. These powerful techniques make analysis of alternating currents much easier. The exercises in this 'lab' do not require equipment; they are traditional homework problems.

6.1 Resistors, Capacitors, and Inductors

Resistance is commonly denoted by the letter \( R \); the SI unit for resistance is the ohm \( (\Omega \equiv \text{volt/amp}) \). To cause a current \( I \) (SI unit: amp \( \equiv \text{coul/s} \)) to flow through a resistor, you must apply a voltage or potential (SI unit: volt \( \equiv J/\text{coul} \)). This is governed by Ohm's law:

\[
V_R = IR
\]  

\( (6.1) \)

Capacitance is commonly denoted by the letter \( C \); the SI unit for capacitance is the farad \( (F \equiv \text{coul/volt}) \). If you apply a voltage to the capacitor, you can cause a charge \( Q \) to accumulate according to

\[
V_C = \frac{Q}{C}
\]  

\( (6.2) \)

This really means that you cause a current to flow, which charges up the capacitor. Since current is charge per time that flows into the capacitor, the accumulated charge is simply

\[
Q = \int 1dt
\]  

\( (6.3) \)

Inductance is commonly denoted by the letter \( L \); the SI unit for inductance is the henry \( (H \equiv \text{volt} \cdot \text{s/amp}) \). An inductor is essentially a coil of wire that produces a magnetic field when current flows through it. It takes energy to establish magnetic fields (returned when the field turns off), which causes inductors to oppose
changes in current. The voltage across an inductor is related to the change in current flowing through it according to

\[ V_L = L \frac{dI}{dt} \tag{6.4} \]

### 6.2 AC Driven Series LRC Circuit

Now consider a series circuit comprised of a resistor \( R \), a capacitor \( C \), and an inductor \( L \), as depicted in Fig. 6.1. If you apply a voltage \( V \) to the whole circuit, it will be balanced by the sum of the voltages across the individual elements:\(^1\)

\[ V = V_R + V_C + V_L \tag{6.5} \]

Substitution of (6.1)-(6.4) into (6.4) yields

\[ V = RI + \frac{1}{C} \int I \, dt + L \frac{dI}{dt} \tag{6.6} \]

Now suppose that your applied voltage is a sinusoidal waveform

\[ V(t) = V_0 \cos \omega t \tag{6.7} \]

You might like to know what current \( I(t) \) flows in the circuit as a result of the applied voltage. In this case, you will be interested to know that the solution to (6.5) has the form

\[ I(t) = I_0 \cos (\omega t - \phi) \tag{6.8} \]

The current also oscillates with frequency \( \omega \). The amplitude \( I_0 \) and phase \(^2\) \( \phi \) both depend on \( R, C, L \), and \( \omega \). A different phase means that the oscillating current is shifted in time relative to the applied voltage, as depicted in Fig. 6.2.

It turns out that finding \( I_0 \) and \( \phi \) is a bit of work. Since it involves the cosine, not surprisingly, there is trigonometry involved. If you are interested, the solution is worked out for you using brute force in appendix 6.A, but first we will show you a much better way. (You might want to glance briefly at appendix 6.A to better appreciate the alternative.) It turns out that complex numbers come in handy for managing the trigonometry involved in solving sinusoidally driven LRC circuits. The more complicated the circuit, the more powerful the technique becomes. Because complex-number notation is used so pervasively in physics, it is worth making a significant investment at this time.

---

\(^1\)It doesn't matter what order you add these voltages up, so you can hook up the elements in any order.

\(^2\)We have followed the convention of the Physics 220 textbook, where the phase \( \phi \) is introduced with a minus sign. Different conventions do not change the physics, although they have the potential to lead to confusion.

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6.3 Complex Numbers

As we saw in the previous section, AC circuits involve functions of the form \( V = V_0 \cos \omega t \) and \( I = I_0 \cos (\omega t - \phi_I) \), where \( V \) and \( I \) represent voltage and current. The phase term \( \phi \) in the cosine means that the sine function is also intrinsically present through the identity

\[
\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \tag{6.9}
\]

With a good background in trigonometry, one can solve AC-circuit problems. However, (6.9) and similar trig identities can be cumbersome to use. Fortunately, complex notation offers an equivalent approach that is far more convenient. The modest investment needed to become comfortable with complex notation will definitely be worth it.

Hopefully, you have encountered complex numbers before, where \( i \equiv \sqrt{-1} \). Generically, a complex number is written in the form \( z = a + ib \), where \( a \) and \( b \) are real numbers. The real part, denoted \( \text{Re}\{z\} \), is \( a \), and the imaginary part, denoted \( \text{Im}\{z\} \), is \( b \). As an example, if \( z = 2 - i5 \), then \( \text{Re}\{z\} = 2 \) and \( \text{Im}\{z\} = -5 \).

A common way of obtaining the real part of an expression is simply by adding the complex conjugate and dividing the result by 2:

\[
\text{Re}\{z\} = \frac{1}{2} (z + z^*) = \frac{1}{2} (a + ib) + \frac{1}{2} (a - ib) = a \tag{6.10}
\]

The complex conjugate of \( z = a + ib \), denoted with an asterisk, is \( z^* = a - ib \). A number times its own complex conjugate is guaranteed to be real and non-negative:

\[
z^* z = (a - ib)(a + ib) = a^2 - iab + iab + b^2 = a^2 + b^2 \tag{6.11}
\]

The complex conjugate is useful for eliminating complex numbers from the denominator of expressions. Multiply and divide any expression by the complex conjugate of the denominator:

\[
\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \tag{6.12}
\]

The absolute value (sometimes called modulus or magnitude) of a complex number is the square root of this product:

\[
|z| \equiv \sqrt{z^* z} = \sqrt{a^2 + b^2} \tag{6.13}
\]

6.4 Euler's Equation

So how do complex numbers relate to trigonometric functions? The connection is embodied in Euler's formula:

\[
e^{i\alpha} = \cos \alpha + i \sin \alpha \tag{6.14}
\]
Euler's formula can be proven using a Taylor's series expansion (about the origin):

\[ f(x) = f(0) + \frac{x}{1!} \frac{df}{dx} \bigg|_{x=0} + \frac{x^2}{2!} \frac{d^2 f}{dx^2} \bigg|_{x=0} + \frac{x^3}{3!} \frac{d^3 f}{dx^3} \bigg|_{x=0} + \ldots \]  

(6.15)

Expanding each function appearing in (6.14) about the origin yields

\[ \cos \alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots \]  

(6.16)

\[ i \sin \alpha = i \alpha - \frac{i \alpha^3}{3!} + i \frac{\alpha^5}{5!} - \cdots \]  

(6.17)

\[ e^{i\alpha} = 1 + i \alpha - \frac{\alpha^2}{2!} - i \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + i \frac{\alpha^5}{5!} - \cdots \]  

(6.18)

One can quickly verify that (6.18) is the sum of (6.16) and (6.17), which completes the proof.

By inverting Euler's formula (6.14) (i.e. use algebra on the formula written with \( \alpha \) and again with \(-\alpha\)), we obtain the following representation of the cosine and sine:

\[ \cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \quad \text{and} \quad \sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \]  

(6.19)

**Example 6.1**

Using (6.19), prove (6.9).

**Solution:**

\[ \cos \alpha \cos \beta + \sin \alpha \sin \beta = \frac{e^{i\alpha} + e^{-i\alpha}}{2} e^{i\beta} + \frac{e^{i\alpha} - e^{-i\alpha}}{2i} e^{-i\beta} \]

\[ = \frac{e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{-i(\alpha+\beta)} + e^{-i(\alpha-\beta)}}{4} \]

\[ = \frac{e^{i(\alpha+\beta)} + e^{-i(\alpha-\beta)}}{2} = \cos(\alpha - \beta) \]

**6.5 Polar Format of Complex Numbers**

With the aid of Euler's formula, it is possible to transform any complex number \( z = a + ib \) into the form \( \rho e^{i\phi} \) where \( a, b, \rho, \) and \( \phi \) are all real. From (6.19), we see that the required connection between \( (\rho, \phi) \) and \( (a, b) \) is

\[ \rho e^{i\phi} = \rho \cos \phi + i \rho \sin \phi = a + ib \]  

(6.20)

The real and imaginary parts of this equation must separately be equal, which forces \( a = \rho \cos \phi \) and \( b = \rho \sin \phi \). These two equations can be solved together to yield

\[ \rho = \sqrt{a^2 + b^2} = |z| \quad \text{and} \quad \phi = \tan^{-1} \left( \frac{b}{a} \right) \quad (a > 0) \]  

(6.21)
6.6 Real vs Complex AC Quantities

Note that \( \rho \) is just the absolute value of a complex number as defined in (6.13).

The transformations (6.21) have a clear geometrical interpretation in the complex plane, and this makes them easier to remember. They are just the usual connections between Cartesian and polar coordinates. As seen in Fig. 6.3, \( \rho \) is the hypotenuse of a right triangle having legs with lengths \( a \) and \( b \), and \( \phi \) is the angle that the hypotenuse makes with the \( x \)-axis.

**Example 6.2**
Write \(-3 + 4i\) in polar format.

**Solution:** We must be careful with the negative real part since it indicates quadrant III, which is outside of the domain of the inverse tangent (quadrants I and IV). Best to factor the negative out and deal with it separately:

\[ -3 + 4i = -(3 + 4i) = -\sqrt{3^2 + (-4)^2} e^{i\tan^{-1}\frac{-4}{3}} = e^{i\pi} 5 e^{-i\pi} = 5 e^{i\pi} \]

Here, we used the fact that \( e^{i\pi} = \cos \pi + i \sin \pi = -1 \).

The phase \( \phi \) in \( e^{i\phi} \) is assumed to be expressed in radians (rather than degrees) unless otherwise specified, which is uncommon.

**6.6 Real vs Complex AC Quantities**

Here are a couple of tricks worth mentioning.

**Hiding the Phase in a Complex Amplitude**
An AC quantity like current can either be written as a real function, \( I_0 \cos(\omega t - \phi_I) \), or as the real part of a complex function, \( \tilde{I}_0 e^{i\omega t} \), where the phase \( \phi_I \) is conveniently hidden in the complex factor \( \tilde{I}_0 \equiv I_0 e^{-i\phi_I} \), where \( I_0 \) is real. These two expressions can be reconciled using Euler’s formula (6.14), showing that

\[ I_0 \cos(\omega t - \phi_I) = \text{Re}\{ \tilde{I}_0 e^{i\omega t} \} \] (6.22)

**Fast and Loose with \( \text{Re}\{\} \)**
As a reminder, the operation \( \text{Re}\{\} \) retains only the real part of the argument without regard for the imaginary part. It is common (even conventional) in physics to omit the explicit writing of \( \text{Re}\{\} \). Thus, physicists participate in a secret conspiracy (don’t let the mathematicians find out) where \( \tilde{I}_0 e^{i\omega t} \) actually means \( I_0 \cos(\omega t - \phi_I) \). This laziness is permissible because it is possible to perform linear operations such as addition, differentiation, or integration while procrastinating the taking of the real part until the end. That is

\[ \text{We will not face this situation in LRC circuits because resistance is non negative.} \]
6.7 LRC Circuits in Complex Notation

We return to the AC driven circuit considered previously in section 6.2. We write the applied voltage (6.7) as

\[ V(t) = V_0 e^{i\omega t} \quad (6.25) \]

In keeping with our conspiracy, we don’t bother writing explicitly \( \text{Re}\{\}\). Secretly, we really mean \( \text{Re}\{V(t)\} \), which according to (6.14) is just \( V_0 \cos \omega t \).

We expect the current flowing in the circuit to oscillate at the same frequency \( \omega \), so we write

\[ I(t) = \tilde{I}_0 e^{i\omega t} \quad (6.26) \]

In this case, \( \tilde{I}_0 \) must be regarded as a complex number to account for the fact that it can oscillate with a phase different from the applied voltage. As before, taking the real part is implied, so when you look at (6.26), you should be thinking (6.22).

When we plug the above voltage and current into our series AC circuit equation (6.6), \( V = RI + \frac{1}{C} \int I dt + \frac{L}{d} \frac{dI}{dt} \), we get

\[ V_0 e^{i\omega t} = R \tilde{I}_0 e^{i\omega t} + \frac{1}{C} \int \tilde{I}_0 e^{i\omega t} dt + \frac{L}{d} \frac{d}{dt} \tilde{I}_0 e^{i\omega t} = \tilde{I}_0 e^{i\omega t} \left( R + \frac{1}{i\omega C} + i\omega L \right) \quad (6.27) \]

Therefore,

\[ \tilde{I}_0 = \frac{V_0}{R + \frac{1}{i\omega C} + i\omega L} \quad (6.28) \]

We are essentially done. We have the solution for the current. The only thing left to do is to rewrite the complex number in the more convenient polar format before taking its real part. (Only the real part is physical.)
Example 6.3

Find the amplitude and phase of the current in a series LRC circuit driven with AC voltage having peak amplitude 10 V and frequency \( f = 60 \text{Hz} \), if \( R = 50 \Omega \), \( C = 60 \mu \text{F} \), and \( L = 40 \text{mH} \).

Solution: The angular frequency is \( \omega = 2\pi f = 2\pi 60 \text{s}^{-1} = 377 \text{s}^{-1} \). The complex current amplitude (6.27) is

\[
\tilde{I}_0 = \frac{10 \text{ V}}{50 \Omega + \frac{1}{i(377 \text{s}^{-1}) (6\times10^{-5} \text{F})} + i (377 \text{s}^{-1}) (4 \times 10^{-2} \text{H})} = \frac{10 \text{ V}}{(50 - i29.1) \Omega} = 0.173 \text{A} e^{i0.527}
\]

The overall current is then

\[
I(t) = \text{Re}\{\tilde{I}_0 e^{i\omega t}\} = I_0 \cos(\omega t - \phi) = 173 \text{ mA} \cos\left(377 \text{s}^{-1} t - (-0.527)\right)
\]

Because \( \phi \) is negative, the current leads the voltage by (0.527 rad) \( (180^\circ / (\pi \text{ rad})) = 30.2^\circ \).

6.8 Equipment

No equipment needed.

Appendix 6.A  Solving an LRC Circuit the Hard Way: with Sine and Cosine

If you read this appendix, you are a real nerd, but here goes. We solve the series LRC circuit (6.6), \( V = RI + \frac{1}{C} \int i \text{d}t + L \frac{di}{dt} \), without complex notation. As usual, we assume the circuit is driven with voltage \( V = V_0 \cos \omega t \). Since we expect the current to oscillate with the same frequency but at a different phase, we need to allow the solution to include both cosine and a sine:

\[
I(t) = A \cos \omega t + B \sin \omega t \tag{6.29}
\]

We need to plug (6.29) into (6.6) to see 1) if it works, and, if it does, 2) what \( A \) and \( B \) should be. Plugging in, we get

\[
V = R (A \cos \omega t + B \sin \omega t) + \frac{1}{C} \int (A \cos \omega t + B \sin \omega t) \text{d}t + L \frac{d}{dt} (A \cos \omega t + B \sin \omega t)
\]

or

\[
V_0 \cos \omega t = R (A \cos \omega t + B \sin \omega t) + \frac{1}{\omega C} (A \sin \omega t - B \cos \omega t) + \omega L (-A \sin \omega t + B \cos \omega t)
\]

For this equation to be true, we need the coefficients on the cosine and the sine to separately balance. The requirement is

\[
V_0 = RA - B \frac{1}{\omega C} + \omega LB \quad \text{and} \quad RB + \frac{A}{\omega C} - A \omega L = 0 \tag{6.30}
\]
Now for a little algebra on these two equations. From the latter equation, we have
\[ B = \frac{A}{R} \left( \omega L - \frac{1}{\omega C} \right), \]
which we plug into the former equation to find
\[
A = \frac{V_0 R}{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2},
\]
and, after back substituting,
\[
B = \frac{V_0 \left( \omega L - \frac{1}{\omega C} \right)}{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}.
\]

We can put our trial solution (6.29) into the form (6.8) by factoring out \( \sqrt{A^2 + B^2} \) as follows:
\[
I(t) = \sqrt{A^2 + B^2} \left( \frac{A}{\sqrt{A^2 + B^2}} \cos \omega t + \frac{B}{\sqrt{A^2 + B^2}} \sin \omega t \right),
\]
(6.33)

Referring to Fig. 6.5, we can think of \( A/\sqrt{A^2 + B^2} \) and \( B/\sqrt{A^2 + B^2} \) as the cosine and sine of an angle
\[
\phi_I = \tan^{-1} \left( \frac{B}{A} \right) = \tan^{-1} \left( \frac{\omega L - \frac{1}{\omega C}}{R} \right),
\]
(6.34)

Then we may write
\[
I(t) = \sqrt{A^2 + B^2} \left( \cos \phi_I \cos \omega t + \sin \phi_I \sin \omega t \right) = \sqrt{A^2 + B^2} \cos \left( \omega t - \phi_I \right),
\]
(6.35)

where we have used the angle-addition formula for cosine. To make this match the form (6.8), \( I(t) = I_0 \cos \left( \omega t - \phi_I \right) \), we assign
\[
I_0 = \sqrt{A^2 + B^2} = \frac{V_0}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}},
\]
(6.36)
Exercises

A. Complex Arithmetic

L6.1  Compute
(a) \((-1 + 2.7i) + (13.9 - 0.5i)\)
(b) \((\sqrt{3} + 8i) + (-2\sqrt{3} - 7i)\)
(c) \(\frac{z + z^2}{2}\), where \(z = 2 - 3i\)
Example: \((5 + 3i) + (-2 + 7i) = 3 + 10i\)

L6.2  Compute
(a) \(i(-1 + i)\)
(b) \((3 + 4i)(3 - 4i)/25\)
(c) \(1 + i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7\)
Example: \((2 + 4i)(3 - 2i) = 6 - 4i + 12i + 8 = 14 + 8i\)

L6.3  Compute
(a) \(1/i\)
(b) \(2/(1 - i)\)
(c) \((5.2 - 4.8i)/(0.6 + 0.8i)\)
Example: \((5 + 10i)/(1 - 2i) = \frac{(5 + 10i)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \frac{-15 + 20i}{5} = -3 + 4i\)

L6.4  Compute
(a) \(|i|\)
(b) \(|4 + 3i|\)
(c) \(|\cos \alpha + i \sin \alpha|\)
Example: \(|3 + 4i| = \sqrt{3^2 + 4^2} = 5\)

L6.5  Convert to polar format:
(a) \(-i\)
(b) \(5.71 - \sqrt{3}i\)
(c) \(-3 - 2i\)
Example: \(3 - 3i = \sqrt{3^2 + (-3)^2} e^{i \tan^{-1}(-3/3)} = 3\sqrt{2} e^{-i\pi/4}\)

L6.6  Convert to standard form:
(a) \(2 e^{i\pi/6}\)
(b) \(8.9 e^{-100i}\) (Be careful: 100 expresses radians – not degrees.)
(c) \(e^{1+i2}\)
Example: \(4.5 e^{-1.2i} = 4.5 [\cos (-1.2) + i \sin (-1.2)] = 1.63 - 4.19i\)

B. Complex Functions

L6.7  Which is not true?
(a) The real part of the sum of two functions is the same as the sum of the real parts of the functions.
(b) The real part of the derivative of a function is the same as the derivative of the real part.
(c) The real part of the integral of a function is the same as the integral of the real part.
(d) The real part of the product of two functions is the same as the product of the real parts of the functions.

L6.8  Recast \( u = 1 + \frac{i}{\omega} \) into the form \( u = |u| e^{i\phi} \), assuming \( \omega \) is a real. Plot both \(|u|\) and \(\phi\) as functions of \(\omega\) over the range -3 to 3. You may use a computer to graph the curves or draw them by hand.

L6.9  Recast \( v = 2e^{i\omega} \), where \(\omega\) is real, into standard form \(\text{Re}\{v\} + i\text{Im}\{v\}\). Plot both \(\text{Re}\{v\}\) and \(\text{Im}\{v\}\) as functions of \(\omega\) over the range \(-\pi\) to \(\pi\). You may use a computer to graph the curves or draw them by hand.

L6.10 Following the method used in example 6.1, prove
\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha
\]

L6.11 Following the method used in example 6.1, prove
\[
\cos^2 \alpha + \sin^2 \alpha = 1
\]

C. Series AC Circuits

L6.12 Show that \(\frac{1}{(\omega C)}\) and \(\omega L\) both have units of ohms (\(\Omega\)). (See section 6.1.)

L6.13 The LRC circuit in Example 6.3 (\(R = 50 \Omega, C = 60 \mu F, L = 40 \text{ mH}\)) was driven at \(f = 60 \text{ Hz}\). Plot voltage and the current as functions of time on the same graph over a period of two oscillations. Don't worry about amplitudes and units on the vertical axis; but be sure to accurately portray the relative phase.

L6.14 Repeat the previous exercise for \(f = 600 \text{ Hz}\). You will need to recompute the phase of the current at this frequency.

L6.15 At what frequency will the voltage and current in Example 6.3 be in phase? Hint: Set \(\phi = 0\) and solve for \(f\).
Lab 7

AC Impedance and High-Pass Filters

In this lab you will become familiar with the concept of AC impedance and apply it to the frequency response of a high-pass filter.

7.1 AC Impedance

Just as Ohm’s law $V = IR$ applies to DC circuits, a similar law applies to AC circuits:

$$V(t) = I(t) Z$$  \hspace{1cm} \text{(Ohm’s law for AC circuits)} \hspace{1cm} (7.1)$$

where $Z$ is a complex quantity called impedance. The impedance has a real and an imaginary part. The real part is called the resistance and the imaginary part is called the reactance. Resistance, reactance, and impedance all have the same units of ohms.

$$Z = \frac{V(t)}{I(t)} = \frac{V_0 e^{i\omega t}}{I_0 e^{i\omega t}} = \frac{V_0}{I_0}$$ \hspace{1cm} (7.2)$$

For the case of a series LRC circuit such as shown in Fig. 7.1, (6.28) reveals that

$$Z = R + \frac{1}{i\omega C} + i\omega L = |Z| e^{i\phi}$$ \hspace{1cm} \text{(series circuit)} \hspace{1cm} (7.3)$$

You can work out using (6.21) that

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$ \hspace{1cm} \text{(series circuit)} \hspace{1cm} (7.4)$$

In fact, we can think of the overall impedance in our series AC circuit as being comprised of three elements:

$$Z = Z_R + Z_C + Z_L \quad \text{where} \quad Z_R \equiv R, \quad Z_C \equiv -\frac{i}{\omega C} \quad \text{and} \quad Z_L \equiv i\omega L$$ \hspace{1cm} (7.5)$$
Thus, the impedances add together like three resistors in a series circuit. Even if you feel a little shaky about the behaviors of resistors, inductors, and capacitors and AC circuits, you will need to memorize these three impedance expressions and be able to use them.

In the introductory electronics lab course (Physics 140), you learned how to add resistors in series and in parallel:

\[
R_{\text{series}} = R_1 + R_2 \quad \text{and} \quad \frac{1}{R_{\text{parallel}}} = \frac{1}{R_1} + \frac{1}{R_2}
\]

(7.6)

It turns out that same rules apply to impedance (although above we looked only at a series circuit):

\[
Z_{\text{series}} = Z_1 + Z_2 \quad \text{and} \quad \frac{1}{Z_{\text{parallel}}} = \frac{1}{Z_1} + \frac{1}{Z_2}
\]

(7.7)

This is great news! The lumped behavior of any complicated network of resistors, inductors, and capacitors can be described by the total impedance that is built up from individual components using familiar series and parallel addition.

### 7.2 ELI the ICE Man

From (7.2), we see that the peak magnitude of the current oscillations in a circuit is

\[
I_0 = |I_0| = \frac{V_0}{|Z|}
\]

(7.8)

Because the voltage across a resistor is always in phase with the current through the resistor, the real-valued quantity \(Z_R\) is just the usual resistance. Notice, however, that \(Z_C\) and \(Z_L\) are purely imaginary-valued quantities that comprise the reactive (rather than resistive) part of the impedance. The reactance generally depends on the frequency (\(\omega\)) and is the origin of the phase difference between the current and voltage.

For an inductive reactance, the current lags the voltage by 90°, dictated by the factor of \(i = e^{i\pi/2} = e^{i(90°)}\) in \(Z_L\). If only an inductor is present, then \(Z = \omega L e^{i\pi/2}\), in the denominator of (7.2), causes \(i\pi/2\) to be subtracted from \(i\omega t\), which causes the current to oscillate a quarter cycle behind the voltage. For a capacitive reactance, the current leads the voltage by 90°, because of the factor \(-i = e^{-i\pi/2} = e^{i(-90°)}\) in \(Z_C\). This is something you can easily see on an oscilloscope, even if you have all three elements \(L,R,C\) in the circuit at once. To see the current, just measure the voltage across a resistor, since for the resistor, the current and voltage are in phase. Then look simultaneously at the voltage across either the capacitor or the inductor.

It is easy to get mixed up about whether current leads or lags the voltage. The mnemonic “Eli the Ice man” might help you remember it. As usual \(L\) stands for inductor, \(C\) stands for capacitor, and \(I\) stands for current. Voltage is represented by \(E\). ELI reminds us that voltage comes before current for (mostly) inductive circuit (i.e. to the left on an oscilloscope trace, which corresponds to earlier time), and ICE reminds us that current comes before voltage for a (mostly) capacitive
circuit. While this is always true, be aware that different conventions that are used have potential to mix you up. Best not to memorize this based on the sign of the phase, because that would depend whether the phase is placed on the voltage or on the current, or perhaps even defined with a built-in sign (as is done here and in the Physics 220 textbook).

### 7.3 AC Gain

If you think of the AC source as a voltage input with amplitude $V_{\text{in}}$, and the potential difference across some portion of your circuit as a voltage output, it is intuitive to define the voltage gain as $G = V_{\text{out}} / V_{\text{in}}$. After choosing the portion of the circuit that forms the output, and determining the impedance of that element, $Z_{\text{out}}$, one can compute the expected gain of a circuit as

$$G = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{I Z_{\text{out}}}{V} = \frac{Z_{\text{out}}}{Z}$$

(7.9)

where $Z$ is the total impedance of the circuit as seen by the source. The word ‘gain’ is used loosely here since the passive circuits that we consider have a gain less than one (i.e., they can only absorb energy), as opposed to an amplifying circuit.

**Example 7.1**

Find the gain of a series LRC circuit as shown in Fig. 7.1, if the output is considered to be the voltage across the resistor.

**Solution:** We have $Z_{\text{out}} = R$ and from (7.3) $Z = R - \frac{1}{\omega C} + i \omega L$. The gain is then

$$G = \frac{R}{R + i \left( \omega L - \frac{1}{\omega C} \right)}$$

As outlined in section 6.5, it is helpful to change the complex number in the denominator to polar format:

$$G = |G| e^{-i\phi} \quad \text{where} \quad |G| = \frac{R}{\sqrt{R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2}} \quad \text{and} \quad \phi = \tan^{-1}\left( \frac{\omega L - \frac{1}{\omega C}}{R} \right)$$

(7.10)

### 7.4 RC Filter

A filter has a frequency-dependent gain that is designed to pass certain frequencies and block others. It has a high gain at frequencies that should be passed (i.e., output comparable to input) and very low gain at frequencies that should be blocked. A simple high-pass filter can be constructed by setting $L = 0$, which

![Figure 7.2 Frequency response of an RC 'high-pass' filter.](image)
makes an RC circuit. From (7.10), the magnitude of the gain across the resistor simplifies in this case to

\[ |G| = \frac{R}{\sqrt{R^2 + 1/((\omega C)^2)}} \quad \text{(high-pass filter)} \quad (7.11) \]

A graph of \(|G|\) as a function of frequency is shown in Fig. 7.2. As can be seen, the voltage across the resistor is virtually nonexistent for very low frequencies and then shoots up to a value near one as the frequency is cranked up.

The cutoff frequency \(f_0\) marks the transition between the ‘block’ and ‘pass’ regions of the frequency spectrum. We define the cutoff frequency as the place where the gain equals \(1/\sqrt{2} = 0.707\). Setting the gain expression equal to this value, we solve for \(f_0\) as follows:

\[ \frac{R}{\sqrt{R^2 + 1/(2\pi f_0 C)^2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad 2R^2 = R^2 + \frac{1}{(2\pi f_0 C)^2} \quad \Rightarrow \quad f_0 = \frac{1}{2\pi RC} \quad \text{(high-pass filter)} \quad (7.12) \]

### 7.5 Equipment

LRC component board, LC meter, signal generator/oscilloscope stack, frequency meter, amplifier, cables.
Quiz

Q7.1  The "ELI" mnemonic refers to an LRC circuit in which
(a) the voltage leads the current in time.
(b) the current leads the voltage in time.
(c) the current and voltage are in phase.

Q7.2  The “ICE” mnemonic refers to an LRC circuit in which
(a) the inductor has more influence that the capacitor.
(b) the capacitor has more influence than the inductor.
(c) the capacitor and inductor are balanced.
(d) the resistor dominates both the inductor and the capacitor.

Q7.3  Consider an RC circuit with $R = 40\,\Omega$, $C = 50.0\mu F$ driven by an AC source voltage at frequency $f = 160\,\text{Hz}$. Compute the following quantities.
(a) $Z_R$
(b) $Z_C$
(c) $\text{Re}[Z]$
(d) $\text{Im}[Z]$
(e) $|Z|$
(f) $\phi$ (in degrees)
(g) $|G|$ (output measured across $R$).

Q7.4  For the RC high-pass filter in the previous question, compute the cutoff frequency $f_0$ (in Hz).
Exercises

A. Compute the frequency response of a high-pass RC filter circuit.

L7.1 For the high-pass RC filter shown in Fig. 7.3, find expressions for the following quantities in terms of $R$, $C$, and $\omega$:

(a) $\text{Re}\{Z\}$
(b) $\text{Im}\{Z\}$
(c) $|Z|$
(d) $\phi$ such that $Z = |Z| e^{i\phi}$
(e) $|G|$

L7.2 Use the Mathematica routine provided (impedance.nb) to repeat these calculations. Compare them to your manually-computed results to make sure that both are correct.

L7.3 Using the components provided in the lab, design a high-pass RC filter which has a transition frequency somewhere between $f_0 = 1.0$ kHz and 6.0 kHz. Use an R/L/C multimeter to check the actual values of each of your components, which may not agree with the values on the printed labels. Enter the values from your circuit in impedance.nb, and plot the gain from 10 Hz to 100 kHz. If you define $|G|$ properly, it should approach a value of zero at one frequency extreme and a value of one at the other. Paste your plots into your lab notebook and discuss them with your TA and/or lab partner to ensure that they are correct and make intuitive sense.

B. Measure and analyze a high-pass RC filter circuit.

L7.4 Prepare to drive and measure the response of your circuit. Don’t hook it up yet. Send a sine wave signal from the “High” output of your signal generator to your power amplifier (via the line input) and from there to the channel 1 input of your oscilloscope. Set your scope to trigger externally and obtain the trigger signal from the TTL output of your signal generator. Make sure that you have the DC bias of your signal generator “OFF” in order to avoid overheating your amplifier! This amplifier really hates a DC bias. Adjust your signal generator gain to the middle of its range, and then adjust the amplifier gain until the peak-to-peak signal from the power amplifier is 5 V. Use the frequency counter to obtain an accurate measure of your signal frequency. The ground leads from each device must all be connected to each other. Have your TA review your progress before proceeding.

L7.5 You will now hook your circuit up using banana cables to drive it with the 5V signal from your amplifier. (We call this the input signal.). Let the resistor in your circuit be the output element and send its voltage
to channel 2 of your oscilloscope. Simultaneously view the input and output signals on your scope. Sweep the entire frequency range from 0 to 100 kHz to get a qualitative feel for the behavior of the gain (i.e. the output to input ratio). *Beware that you may see a frequency dependence in the input as well as the output voltage, which you will have to take into account.*

**L7.6** Map out the frequency-dependent gain of your circuit from 100 Hz to 100 kHz in 30 uniform logarithmic increments of \( \log_{10}(f) \): Define \( a = \log_{10}(10^2) = 2 \) and \( b = \log_{10}(10^5) = 5 \), use Excel to make a list of \( x \) values stretching from 2 to 5 in 30 uniform linear increments (31 data points), and then calculate \( f = 10^x \) for each value in the list. Measure and record the magnitude \( |G| \) of the gain at each frequency. Also estimate the approximate error in your gain measurements. Use a frequency meter across the input in order to accurately measure frequency.

**L7.7** Create a three-column dataset in Excel. Name the columns “Frequency (Hz)”, “Gain” and “sigmaG”. Using the NLSQfit.nb Mathematica notebook, fit a model of the form \( |G| = A/\sqrt{1 + 1/(2\pi f RC)^2} \) to your data, using \( A \) and \( C \) as variables, but fixing \( R \) to the value measured using a multimeter. Note that \( A \) is an overall scale factor that compensates for the fact that your gain curve may not peak at exactly one. Display the data and the fitted curve on the same graph, using open circles with error bars for the data and a horizontal log scale. *If the error bars are properly estimated, the fitted curve should mostly stay within the error bars of the individual data points.* Place the final graph in your lab notebook, and be sure to record and interpret the fitting parameters and their uncertainties. Comment on how your fitted capacitance compares to the measured capacitance? Was the fitted value within one estimated standard deviation (i.e. the uncertainty from the fit) of the measured value?

**L7.8** Briefly look through the whole frequency range again to observe the behavior of the phase of the output voltage relative to the input voltage and comment. *Keep in mind that time runs from left to right across the oscilloscope screen.* Note that the phase must stay within the range between \(-90^\circ\) and \(+90^\circ\). A \( 90^\circ \) phase corresponds to an input that peaks right where the output is rising through zero. Make a qualitative graph of the frequency-dependent phase for your lab notebook.
Lab 8

Time and Frequency Responses of Low-Pass and High-Pass Filters

In this lab you will apply the concept of AC impedance to the frequency response of a low-pass filter, and explore the relationship between frequency response and time response.

8.1 LR Filter

A simple low-pass filter can be made from an inductor and a resistor as shown in Fig. 8.1. In this configuration, the gain is defined to be \( G \equiv V_o / V_{in} \). The magnitude of the gain works out to be (see example 7.1 with \( C \to \infty \))

\[
|G| = \frac{R}{\sqrt{R^2 + (\omega L)^2}} \quad \text{(low-pass filter) (8.1)}
\]

which exhibits the frequency dependence shown in Fig. 8.2. Only low-frequency signals are able to pass from the input to the output unimpeded; high frequencies are strongly attenuated. Observe that the trend with frequency is reversed from that of the high-pass filter studied in the previous lab.

The cutoff frequency is again defined as the frequency where \(|G| = 1/\sqrt{2} \approx 0.707\). We solve for \( f_0 \) as follows:

\[
\frac{R}{\sqrt{R^2 + (2\pi f_0 L)^2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad 2R^2 = R^2 + (2\pi f_0 L)^2 \quad \Rightarrow \quad f_0 = \frac{R}{2\pi L} \quad \text{(low-pass filter) (8.2)}
\]

Example 8.1

Write the magnitude of the gain (8.1) in terms of \( f \) and \( f_0 \).

Solution: From (8.2), we have \( 2\pi L = R/f_0 \). When this is inserted into (8.1), it
8.2 Decibel Gain

It is not uncommon to express the gain of a circuit using **decibel** units. The decibel gain is defined as

\[
G_{\text{db}} = 10 \log_{10} |G|^2 = 20 \log_{10} |G|
\]

(8.4)

We use \(|G|^2\) in the definition because oft times it is **power** that is of primary interest. The time-averaged power delivered to the resistor is \(|V_R|^2 / (2R) \propto |G|^2\), as will be discussed in the next lab.

An alternative ‘cutoff frequency’ is sometimes expressed as the frequency where the decibel gain drops to \(G_{\text{db}} = -3.0\). This frequency, which we will call \(f_{3\text{db}}\), is very close to \(f_0\), although not precisely the same. \(^1\)

**Example 8.2**

Show that for practical purposes, \(f_{3\text{db}}\) and \(f_0\) are the same frequency.

**Solution:** When \(f = f_{3\text{db}}\), we have

\[-3.0 = 10 \log_{10} |G|^2 \Rightarrow |G|^2 = 10^{-0.3} \approx 0.5012\]

When \(f = f_0\), we have by definition \(|G|^2 = 1/2\), so clearly \(f_{3\text{db}}\) and \(f_0\) are nearly equal. Solving for the exact connection between \(f_{3\text{db}}\) and \(f_0\) yields

\[f_{3\text{db}} = f_0 \sqrt{10^{0.3} - 1} = 0.995 f_0 \approx f_0\]

(8.5)

8.3 Time Response of Circuits

A time response study addresses the significant question, **How does the output vary with time when the input experiences an instantaneous change?** We can be fairly certain that a real circuit will not respond instantaneously. Even a circuit that responds very quickly will take a finite amount of time to complete its response. This is because every electronic device has some capacitance and some inductance with which to store electric and magnetic energy, respectively. The flow of energy to or from a capacitor or inductor takes time.

\(^1\)It is fortuitous that \(10^{0.3}\) is surprisingly close to 2, which is why 3 db is a handy attenuation factor.
Fig. 8.3 illustrates what happens when a *square-wave* voltage is applied to our low-pass LR circuit. Relative to the input, the output voltage measured across the resistor is strongly modified. Because a square wave alternately increases and decreases sharply, we can observe both the *rise time* and the *fall time* in a single experiment. These turn out to be the same for our LR circuit. However, for more complicated instruments, involving amplifiers and such, the rise and fall times can be quite different.

Consider an LR circuit as depicted in Fig. 8.1. From (6.6), the voltage across the circuit is

\[ V = RI + L \frac{dI}{dt} \]  

(8.6)

Suppose a constant voltage \( V_0 \) is applied to the circuit, which eventually causes current \( I_0 = \frac{V_0}{R} \) to flow. If the voltage is suddenly switched off (i.e. \( V = 0 \)) at \( t = 0 \), the equation governing the current thereafter is

\[ L \frac{dI}{dt} = -RI \quad \text{which has the solution} \quad I = I_0 e^{-\frac{R}{L} t} \]  

(8.7)

The voltage across the resistor is then \( V_R = IR = V_0 e^{-\frac{R}{L} t} \), which decays exponentially towards zero in time.

On the other hand, if the applied voltage and current are initially zero before a constant voltage \( V_0 \) is applied at \( t = 0 \), the equation governing the current thereafter is

\[ L \frac{dI}{dt} = V_0 - RI \quad \text{which has the solution} \quad I = \frac{V_0}{R} \left( 1 - e^{-\frac{R}{L} t} \right) \]  

(8.8)

The voltage across the resistor is then \( V_R = V_0 \left( 1 - e^{-\frac{R}{L} t} \right) \), which rises exponentially towards \( V_0 \) in time.

We define the *response time* \( \tau \) to be time required for the output to complete \((1 - e^{-1}) = 63\%\) of the full response to a sudden change in the input.

**Example 8.3**

Find the response time and fall time for an LR circuit (call it \( \tau_{\text{LR}} \)).

**Solution:** From (8.8), we solve for the rise time by setting

\[ 1 - e^{-\frac{R}{L} \tau_{\text{LR}}} = 1 - e^{-1} \]

which manifestly requires

\[ \tau_{\text{LR}} = \frac{L}{R} \]  

(8.9)

Similarly, from (8.7) the change in current during the fall time is \( I_0 - I = I_0 \left( 1 - e^{-\frac{R}{L} t} \right) \), which gives the same characteristic time (8.9).
8.4 Time Response vs Frequency-Response

If a circuit responds well to high frequencies (i.e. a high-pass filter), it will be able to ‘keep up’ with sudden changes in time. However, it will not be able to cope well with long sustained trends. On the other hand, if a circuit responds well to low frequencies (i.e. low-pass filter), it will be capable tracking long sustained trends (epitomized by DC output), but it will smooth over and ‘miss’ abrupt changes.

One can use the frequency response function (FRF) of a circuit to predict its time response function (TRF). The TRF describes the manner in which, for example, a circuit responds to a square step voltage. Ultimately, the TRF and the FRF are just two different ways at looking at the same system. More intuitively, they have a type of inverse or reciprocal relationship.

### Time Frequency Product

The product of the cutoff frequency \( f_0 \) (for an LR circuit) and the response time \( \tau_{RL} \) is independent of the values of \( R \) and \( L \). Multiplying (8.2) and (8.9), we get

\[
\tau_{RL} f_0 = \frac{L}{R} \frac{R}{2\pi L} = \frac{1}{2\pi}
\]

(8.10)

8.5 Equipment

LRC component board, LC meter, signal generator/oscilloscope stack, frequency meter, amplifier, cables.
Q8.1 When a square wave (as opposed to a sinusoidal wave) is applied to the low-pass LR filter, the current is (choose one)
(a) also a square wave.
(b) a sinusoidal wave.
(c) a wave composed of spliced exponential growth and decay curves.
(d) none of the above.

Q8.2 Consider an LR circuit with $R = 40 \Omega$ and $L = 50 \text{ mH}$ that is driven by an AC voltage source at frequency $f = 160 \text{ Hz}$. Compute the following quantities.
(a) $Z_L$
(b) $|Z|$
(c) $\phi$ (in degrees).
(d) $|G|$ (output measured across $R$).

Q8.3 (a) For the LR low-pass filter in the previous question, compute the cutoff frequency $f_0$ (in Hz).
(b) At what frequency does the decibel gain drop to $G_{\text{db}} = -6 \text{ db}$? And to $G_{\text{db}} = -10 \text{ db}$?

Q8.4 For a high-pass RC filter, show that (7.11) can be written $|G| = \frac{1}{\sqrt{1+(f_0/f)^2}}$, where $f_0$ is given by (7.12).
Exercises

A. Measure and analyze the frequency response of a low-pass RL filter.

We anticipate that the exercises in this section will proceed smoothly and quickly based on your previous experience with high-pass filters.

L8.1 Repeat exercise 7.3 for the RL circuit shown in Fig. 8.4.

L8.2 Repeat exercises 7.4-7.5 with the RL circuit.

L8.3 Repeat the measurements of 7.6-7.7 for this low-pass RL filter. When you get to the part of the exercise that involves curve fitting, employ a model of the form $|G| = \frac{A}{\sqrt{1+(2\pi f L/R)^2}}$, with $A$ and $L$ as variables (fix $R$ to its measured value).

L8.4 Using the fitted value of $L$, calculate the cutoff frequency $f_0$ for the low-pass LR filter.

B. Measure the time response of an RL low-pass filter.

L8.5 Send the TTL output of your signal generator to your amplifier input, and use the amplifier output to apply a sharp 5V square-wave signal to your low-pass LR filter circuit and observe the resulting time response with an oscilloscope. Set the gain of the the amplifier knob to 1 and monitor the temperature at the back of the amplifier to avoid accidental overheating. Adjust the period of the square wave to be long enough that the circuit voltage appears to decay completely, but not much longer. Sketch two periods of the output signal in your lab notebook. Adjust the vertical scale and the horizontal sweep time so that a single voltage-decay curve approximately fills the display region. Visually estimate $\tau_{RL}$, combine it with your value for $f_0$, and verify (8.10). Your conclusions should include an intuitive explanation regarding the features of the frequency-response and time-response functions.

C. Measure the time response of an RC high-pass filter.

L8.6 Reconstruct the high-pass RC filter from the previous lab (Fig. 8.5) using the same values for $R$ and $C$, and repeat exercise L8.5 to obtain and estimate for $\tau_{RC}$. Use the value of $f_0$ obtained in the previous lab to demonstrate that $2\pi f_0 \tau_{RC} = 1$. In your conclusions, please compare and contrast the high-pass and low-pass time response curves in terms of their high-frequency and low-frequency features.\footnote{For your information, if initially the applied voltage and current in the RC circuit are zero, and then a constant voltage $V_0$ is switched on at $t = 0$, the voltage equation analogous to (8.8) is $V_0 = RI(t) + \frac{1}{C} \int_0^t I(t') \, dt'$. The solution is $I = \frac{V_0}{R} e^{-\frac{t}{\tau_{RC}}}$, which gives a response time $\tau_{RC} = RC$.}

Figure 8.4 LR circuit.

Figure 8.5 RC circuit.
Lab 9

LRC Circuits

In this lab you will study AC gain and impedance in LRC circuits.

9.1 LRC Resonance

The series LRC oscillator (see Fig. 9.1) is an important case that has myriad applications. Driven by an AC voltage source, the oscillating current carries energy back and forth between the capacitor (electric field energy) and the inductor (magnetic field energy). The only element where energy is permanently lost is the resistor.

As seen in (7.3), the total impedance of the series circuit is

\[ Z = Z_R + Z_L + Z_C = R + \frac{1}{i\omega C} + i\omega L \]

The current in the circuit is

\[ I = \frac{V}{Z} = I_0 e^{-i\phi} e^{i\omega t} \]

which is out of phase with the driving voltage. From example 7.1, the voltage across the resistor compared to the input voltage is

\[ G \equiv \frac{V_R}{V} = \frac{IR}{IZ} = \frac{R}{Z} = |G| e^{-i\phi} \]  

(9.1)

where

\[ |G| = \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \]  

and

\[ \phi = \tan^{-1}\left( \frac{\omega L - \frac{1}{\omega C}}{R} \right) \]  

(9.2)

Fig. 9.2 shows the magnitude of the LRC gain \(|G|\) as a function of frequency. At resonance, the gain has a magnitude of 1 and phase of zero. At all other frequencies, the magnitude of the gain is less than 1. The peak of the gain curve occurs at the resonance frequency. By inspection of (9.2), the maximum of \(|G|\) is achieved when \(\omega L - 1/\omega C = 0\), which gives a resonance frequency of

\[ \omega_0 = 2\pi f_0 = 1/\sqrt{LC} \]  

(9.3)

At the resonance frequency \(\omega = \omega_0\), it is as if there is no capacitor or inductor in the circuit, so that the impedance becomes purely resistive (R) and the gain is \(G = 1\). If \(R = 0\), then the current amplitude at resonance will either 'run away' to
infinity, or more realistically, the circuit will fail in a spectacular display of light and heat.

A good measure of the width of the resonance peak is obtained by setting \(|G| = 1/\sqrt{2}\) and solving for frequency, which yields \(\omega \approx \omega_0 \pm \frac{R}{2L}\). Thus, the width is approximately \(\Delta \omega \approx \frac{R}{L}\) or \(\Delta f \approx \frac{R}{2\pi L}\). The resistance broadens the peak without changing its location.

Many physical systems besides circuits exhibit resonance effects. The crystal shattered by an opera singer, the tall building that sways back and forth during an earthquake, the rattling of a car driving over the top of the ‘wake-up’ grooves cut into the shoulder of an interstate highway – these are all examples of damped driven resonance. Clearly a shattering crystal or a collapsing building could use some more damping to limit the amplitude of oscillations. Unless strategically damped, most dynamical systems will have in common that they respond strongly to certain resonance frequencies.

9.2 RMS Voltage and Current

Since AC voltage and current oscillate, their individual amplitudes time-average to zero. Instead of using an average, it is common to characterize the effective strength of voltage and current by their root-mean-square or rms values. These are written

\[
V_{\text{rms}} \equiv \sqrt{\text{Re}\{V\}^2} = \sqrt{V_0^2 \cos^2 \omega t} = \frac{V_0}{\sqrt{2}} \tag{9.4}
\]

and

\[
I_{\text{rms}} \equiv \sqrt{\text{Re}\{I\}^2} = \sqrt{I_0^2 \cos^2 (\omega t - \phi)} = \frac{I_0}{\sqrt{2}} \tag{9.5}
\]

Though \(\cos \omega t\) time averages to zero, observe that \(\cos^2 \omega t\) is always positive and averages to 1/2. Fig. 9.3 shows the relationship between voltage amplitude \(V_0\), peak-to-peak voltage \(V_{\text{pp}}\), and rms voltage \(V_{\text{rms}}\).

9.3 Power

Voltage has units of energy per charge (i.e. J/coul). The voltage specifies the amount of energy required to move charge between different regions of electric potentials. For example, it takes 1.5 J to move a coulomb of charge from one terminal of a 1.5 V battery to the other. Meanwhile, the current flowing in a circuit has units of charge per time (i.e. coul/s). Voltage multiplied by current gives the energy per time (i.e. J/s) or power associated with the motion of charge in a circuit.

Because only the real part of our complex expressions for voltage and current is physical, it is crucial to take the real parts of \(V\) and \(I\) before multiplying. We compute the power as follows:

\[
P = \text{Re}\{V\} \text{Re}\{I\} = \text{Re}\{V_0 e^{i\omega t}\} \text{Re}\{I_0 e^{i(\omega t - \phi)}\} \tag{9.6}
\]
Eq. (6.10) and (6.19) are handy for taking and manipulating the real parts:

\[ P = V_0 e^{i\omega t} + e^{-i\omega t} I_0 e^{i\omega t} - i\phi + e^{-i\omega t} + i\phi \]

\[ = \frac{V_0 I_0}{2} \left( \frac{e^{2i\omega t} - i\phi + e^{i\omega t} + e^{-i\omega t} + i\phi}{2} \right) \]

\[ = \frac{V_0 I_0}{2} (\cos(2\omega t - \phi) + \cos \phi) \quad (9.7) \]

Since the power in an AC circuit fluctuates in time, it is helpful to compute a time average. The first term in (9.7) oscillates positive and negative and so averages to zero. Therefore, the time average of the power is

\[ \bar{P} = \frac{V_0 I_0}{2} \cos \phi = \frac{V_{\text{rms}} I_{\text{rms}}}{2} \cos \phi \quad (9.8) \]

where in the latter expression we have utilized (9.4) and (9.5).

The factor \( \cos \phi \) is called the power factor. Whereas in a DC circuit the power is simply \( P = V_0 I_0 \), in an AC circuit the phase between the voltage and current matters. Since \( \tan \phi = \frac{\omega L}{\frac{1}{\omega C} R} \) (see (9.2)), we deduce from Fig. 9.4 that

\[ \cos \phi = \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} = \frac{R}{|Z|} \quad (9.9) \]

Inductors and capacitors do not dissipate energy. They can only momentarily store energy. Only a resistor can dissipate energy.

**Example 9.1**

Show using a series LRC circuit that all of the power (9.8) consumed by the circuit is dissipated in the resistor.

**Solution:** Ohm’s law states \( I = \frac{V}{Z} \), which implies \( I_0 = \frac{V_0}{|Z|} \) and \( I_{\text{rms}} = \frac{V_{\text{rms}}}{|Z|} \). Using this and (9.9), the time-averaged power for the circuit (9.8) may be written

\[ \bar{P} = V_{\text{rms}} I_{\text{rms}} \cos \phi = \frac{V_{\text{rms}} I_{\text{rms}} R}{|Z|} = I_{\text{rms}}^2 R \]

On the other hand, the time-averaged power used in the resistor is

\[ \bar{P}_R = \text{Re} [V_r] \text{Re} [I] = \frac{(\text{Re} [V])^2 R}{|Z|} = \frac{I_{\text{rms}}^2 R \cos^2 (\omega t - \phi)}{2} = \frac{I_{\text{rms}}^2 R}{2} = I_{\text{rms}}^2 R \]

where we have used the fact that the cosine squared averages to 1/2. Thus, \( \bar{P}_R = \bar{P} \), meaning all of the power is dissipated in the resistor.

Below we list several distinct but equivalent expressions for average power, any one of which might be the most convenient depending which circuit parameters are known.

\[ \bar{P} = I_{\text{rms}}^2 R = \frac{I_{\text{rms}}^2}{|Z|} \cos \phi = I_{\text{rms}} V_{\text{rms}} \cos \phi = \frac{V_{\text{rms}}^2}{|Z|} \cos \phi = \frac{V_{\text{rms}}^2}{R} \cos^2 \phi \quad (9.10) \]
9.4 Parallel and Series Circuits

As was mentioned in Lab 7, AC impedances follow the same parallel and series addition rules as resistance does for DC circuits.

\[ Z_{\text{series}} = Z_1 + Z_2 \quad \text{and} \quad 1/Z_{\text{parallel}} = 1/Z_1 + 1/Z_2 \]

**Example 9.2**

Find the impedance of the circuit depicted in Fig. 9.5.

**Solution:** The impedance of the parallel-branch portion of the circuit is

\[ \frac{1}{Z_{\text{parallel}}} = \frac{1}{i\omega L} + \frac{1}{1/(i\omega C)} \Rightarrow Z_{\text{parallel}} = \frac{i}{\omega L} - \omega C \]

This impedance is added in series with the resistor, so that

\[ Z = \frac{i}{\omega L} - \omega C + R = \sqrt{R^2 + \left(\frac{1}{\omega L} - \omega C\right)^2} e^{i\tan^{-1}\left[\frac{1}{R\omega L - \omega C}\right]} \]

The resulting gain \(|G| = R/|Z|\) now has a minimum (rather than a maximum) at \(\omega = 1/\sqrt{LC}\). This circuit is called a notch filter because of the ‘hole’ or ‘notch’ in the spectrum near \(\omega_0\). In contrast to the series LRC circuit, the width of the notch is \(\Delta\omega \equiv \frac{1}{RC}\) or \(\Delta f \equiv \frac{1}{2\pi RC}\).

9.5 Equipment

LRC component board, LC meter, signal generator/oscilloscope stack, frequency meter, amplifier, cables.

**Appendix 9.A Time Response of an LRC circuit**

Consider an LRC circuit as depicted in Fig. 9.1. Fig. 9.6 illustrates what happens when the voltage across an LRC circuit is suddenly turned off. From (6.6), the voltage across the circuit is

\[ V = RI + \frac{1}{C} \int I dt + L \frac{dI}{dt} \quad (9.11) \]

If a certain current \(I_0\) is flowing when the voltage is suddenly switched off at \(t = 0\), the equation governing the current thereafter is

\[ \frac{1}{C} \int I dt + L \frac{dI}{dt} = -RI \]

Figure 9.5 AC circuit with parallel inductor and capacitor.

Figure 9.6 Decaying voltage oscillations in an LRC circuit after an applied voltage is suddenly turned off.
with solution

\[ I = I_0 e^{-\frac{R}{2L} t} e^{i t \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} \]  

(9.12)

The voltage across the resistor is \( V_R = IR = R I_0 e^{-\frac{R}{2L} t} e^{i t \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} \), which both oscillates\(^2\) and decays in time as shown in Fig. 9.6.

---

\(^1\)Another valid solution is \( I = I_0 e^{-\frac{R}{2L} t} e^{-i t \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} \), which can be added in certain combinations with (9.11) to produce various other solutions with \( I(t) = I_0 \).

\(^2\)The solution oscillates only if \( \frac{1}{LC} > \left(\frac{R}{2L}\right)^2 \).
Quiz

Q9.1 A cutoff frequency marks the transition between strong and weak current as a function of frequency for
(a) high-pass filters.
(b) low-pass filters.
(c) both (a) and (b).
(d) an LRC circuit.
(e) all of the above.

Q9.2 For the LRC circuit in Fig. 9.1, let $R = 40 \, \Omega$, $C = 50.0 \, \mu F$ and $L = 50 \, mH$ that is driven by an AC voltage source at frequency $f = 160 \, Hz$. Compute the following quantities:
(a) $|Z|$

(b) $\phi$ (in degrees)

(c) $|G|$ with the output measured across $R$.

Q9.3 For the circuit in the previous exercise, compute
(a) the resonance frequency $f_0$.

(b) $|Z|$ when $f = f_0$.

Q9.4 Explain how the values of $R$, $C$, and $L$ affect the location and width of the resonance peak.
Exercises

A. Measure and analyze the frequency response of an LRC resonator.

We anticipate that the exercises in this section will proceed smoothly and quickly based on your previous experience RC and RL circuits.

L9.1 Repeat exercise L7.3 for the LRC circuit shown in Fig. 9.7, where \( f_0 = \frac{1}{2\pi\sqrt{LC}} \) is the resonance frequency. Try several different values of \( R \), given your choice of \( L \) and \( C \), and observe resulting peak shapes. Be sure to choose an intermediate value that yields a peak that is neither too broad to fit in the range of the measurement nor so narrow that few of your measured frequencies trace out the peak.

L9.2 Repeat exercises L7.4-L7.5 for the LRC circuit.

L9.3 Repeat the measurements of L7.6-L7.7 for the LRC circuit. When you get to the part of the exercise that involves curve fitting, employ a model of the form \( |G| = \frac{A}{\sqrt{1+\left(\frac{2\pi f L}{R}\right)-1/(2\pi f RC)}} \), using \( A \) and \( L \) and \( C \) as variables (fix \( R \) to its measured value). Compare the measured and fitted values of \( L \) and \( C \).

L9.4 Using the fitted values of \( L \) and \( C \), compute the expected resonance frequency in Hz, and compare it to the location of the observed resonance peak.

L9.5 Briefly look through the whole frequency range again to observe the behavior of the phase of the output voltage relative to the input voltage and comment. Keep in mind that time runs from left to right across the oscilloscope screen. Note that the phase must stay within the range between \(-90^\circ\) and \(+90^\circ\). A \( 90^\circ \) phase corresponds to an input that peaks right where the output is rising through zero. Make a qualitative graph of the frequency-dependent phase for your lab notebook. (You can use \( \phi = 360/\Delta t \) to verify \( \phi = \pm 90^\circ \) at the endpoints.) Compare with the phase graph predicted in exercise L9.3.

L9.6 (a) Describe how the current in the circuit varies as a function of frequency from one side of the resonance to the other.
(b) What are the magnitudes of the impedance and gain at the resonance frequency? Please comment.
(c) Explain why the phase changes sign at the resonance frequency.

B. Explore a parallel circuit configuration.

L9.7 Reconfigure your circuit to match the notch filter shown in Fig. 9.5 using your same values of \( R \), \( L \), and \( C \). Repeat exercise L9.2, but don’t
Collect any data. Just scan the frequency range and qualitatively graph the frequency dependence of $|G|$. Try several different values of $R$ and describe its effect on the location and width of the spectral ‘hole’. Refer to example 9.2 as needed.

C. Practice using rms quantities and explore power dissipation.

1.9.8 Send a sine-wave voltage with amplitude $\sim 2.5$ V (i.e. 5 V peak to peak) from the signal generator \textit{(with no DC bias!)} into the input of the power amplifier. Use the gain control knob to vary the amplitude of the output signal between $V_0 = 0$ and $V_0 \approx 10$ V (i.e. 7 V rms, 20 V peak to peak) while monitoring the output with your oscilloscope to make sure that the voltage oscillates without clipping. Use (9.10) to estimate the rms voltage needed to deliver an average of 1/4 W of power to a 100 $\Omega$ resistor. Then attach a 100 $\Omega$ resistor rated for 1/4 W and slowly turn up the voltage to this level. Does the resistor get hot? \textit{Be careful not to burn your finger!} If not, continue slowly to turn up the voltage until the resistor is barely too hot to touch \textit{without pain}, but not much hotter. Record $V_{rms}$, $I_{rms}$, and $P$ at this setting. Is the 1/4 W rating appropriate?

![Resistor color codes](image)

**Figure 9.9** Resistor color codes. The power rating is determined by the size of the resistor – not marked except on the package. The most common resistors (small) are rated 1/4 W.

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Lab 10

Computerized Experimentation

After a crash course in the Labview programming language, you will create a data acquisition VI and use it to measure the angular velocity of a spinning motor axel.

10.1 Becoming Familiar with Labview

The Labview software package is a graphical programming tool for computer-assisted instrument control, data acquisition, and data analysis. Automated experimentation and data acquisition constitute a very valuable skill set that can be applied in almost any research setting. Labview provides a graphical programming (i.e. data-flow programming) environment for such work. With that said, the best way to understand what this means is to simply dive in.

10.2 Equipment

Computer equipped with a DAQ card and interface box, mass-spring pendulum, force transducer, Getting Started with Labview manual, DC motor with power supply, laser and photodiode, rods and clamps.
Exercises

A. Complete all of the exercises in the *Getting Started with Labview* manual up through exercises 4-4.

The manual gives very detailed step-by-step instructions. As you gain familiarity with Labview, you will be able to anticipate the steps in the manual and move much more quickly.

L10.1 Complete the exercises in Chapter 1 of *Getting Started with Labview*. Have your TA initial your lab notebook at the end of each chapter – no formal report required. Use the chart in Fig. 10.3.

L10.2 Repeat for Chapter 2.

L10.3 Repeat for Chapter 3.

L10.4 Repeat for pages 4-1 to 4-5, where page 4-5 is subject to the following special requirements:

1) A graph is added to the VI for the purpose of visualizing the data being read in. Right click the graph on the “Front Panel” to see the graph options. Let the “Visible Items” include the “Cursor Palette” and the “Graph Palette”. Become familiar with these tools and use them to accurately measure the pendulum period. You will also find it helpful to turn off the “Y Autoscaling”, after which you can manually set the new limits by using the text tool to change the top and bottom scale values right on the graph itself.

2) Use a force transducer as the input signal for testing your data-acquisition VI. Hang a mass-spring pendulum from the force transducer and set the transducer to its 10-Newton scale. The transducer has two plug adaptors that connect to the computer interface box located on the shelf below your computer, one plug for power (choose any of four power jacks on the interface box) and one plug for analog input (channel 1 on the interface box works with Labview channel ai0). Check to see that the red power light on the interface box is on. Inside the DAQ Assistant, increase the number of samples in order to obtain a measurement time that includes about three oscillation periods.

3) Wire numeric controls to the “Rate and Number of Samples” inputs of the DAQ assistant so that you can more easily experiment with these values. Use the “Custom Scaling” feature of the DAQ Assistant to center the transducer output on 0 V (i.e. remove the DC bias). This will require you to create a new linear scale for which the y intercept is...
equal to your bias voltage. Reduce the signal input range to ±5 V.

B. Measure the rotation frequency of a DC motor axel vs. applied DC voltage.

L10.5 Configure a laser and photodiode so that a piece of tape attached to a motor axle breaks the beam on each rotation. Use the data acquisition VI that you created at the end of part A to read the photodiode voltage into Labview. Sample it rapidly so as to clearly see the time dependence of the rotor position, send the results to a waveform graph, and use the graph cursor tools to obtain an accurate measure of the rotation period. Increment the motor voltage in 1 V steps, record the results in a Microsoft Excel spreadsheet, and plot rotor speed (Hz) as a function of potential difference (Volts). Do not supply more than 20 V to the motor!
In this lab, you will see how the concept of AC impedance can be applied to sinusoidally-driven mechanical and acoustical systems.

11.1 Mechanical Oscillator

Consider the mechanical system shown in Fig. 11.1. Some sort of external force $F_{\text{external}}$ pushes on a mass $m$, which is attached to a spring. Let $x$ be the amount that the spring is stretched from its equilibrium position. The spring imposes a restoring force, which by Hooke’s law is $F_{\text{spring}} = -kx$, where $k$ is called the spring constant. There may also be friction from, for example, air resistance, or perhaps the mass is submerged in a bucket of oil. We assume that the friction is proportional to the velocity of the mass $v$, but in an opposing direction such that $F_{\text{friction}} = -bv$.

This system is called a damped mass-spring oscillator. It is subject to Newton’s third law: $F = ma$. The equation of motion is therefore

$$F = F_{\text{external}} + F_{\text{spring}} + F_{\text{friction}} = ma \quad \Rightarrow \quad F_{\text{external}} = bv + kx + m \frac{dv}{dt} \quad (11.1)$$

where we have used the fact that $a = \frac{dv}{dt}$. This mechanical system is highly relevant to many engineering applications, including acoustics or gently bouncing a baby on a mattress.

The good news is that you already know all about this mechanical system. That is, (11.1) looks just like the equation for a driven series LRC circuit! If we recognize that $x = \int v dt$, then (11.1) can be written as

$$F_{\text{external}} = bv + k \int v dt + m \frac{dv}{dt} \quad (11.2)$$

An analogy with (6.6), $V = RI + \frac{1}{C} \int I dt + L \frac{dI}{dt}$, is now quite clear. The force $F_{\text{external}}$ acts like a voltage, while the velocity $v$ plays the role of current. In this
case, all of the impedance-related concepts and equations for the series LRC circuit may be used. We simply make the substitutions \(L \rightarrow m\), \(\frac{1}{C} \rightarrow k\), and \(R \rightarrow b\). The magnetic field surrounding an inductor is hard to get going and hard to stop, so it acts like inertia (mass). Meanwhile, the electrical field in a capacitor always pushes back like the spring. 1 A resistor dissipates energy just like friction.

If a sinusoidal force \(F_{\text{external}} = F_0 e^{i\omega t}\) is applied, a comparison with (6.26) and (6.28) reveals

\[
v = \frac{F_0 e^{i\omega t}}{b + \frac{k}{ia} + i\omega m} = \frac{F_0 e^{i\omega t}}{\sqrt{b^2 + (\omega m - k/\omega)^2}} e^{-i\phi} \tag{11.3}
\]

where, similar to (7.4) or (7.10), \(\phi = \tan^{-1}\left(\frac{\omega m - k/\omega}{b}\right)\).

The concept of mechanical resonance is immediately apparent. The driving frequency can be chosen to decrease the denominator of (11.3) and thereby maximize the amplitude of oscillations; a well chosen frequency makes a baby on a mattress to go higher.

## 11.2 Helmholtz Resonator

An acoustical resonator, often called a Helmholtz resonator, behaves very similar to a mechanical mass/spring oscillator. It consists of a rigid hollow sphere with a relatively small tube-like opening to the outside. (See Fig. 11.3.) In this case, the compressible gas inside the cavity acts as a spring, while the air within the neck (a tube-shaped opening) acts as the mass. Viscous friction between the air and the walls of the neck introduces damping. The Helmholtz resonator is an especially simple example of an acoustical cavity complete with acoustical resonance. A speaker driver or other sound source just outside the opening excites the resonance inside the cavity.

The resonance frequency of Helmholtz resonator is approximated by

\[
f_0 = \frac{c}{2\pi} \sqrt{\frac{A}{V\ell}} \tag{11.4}
\]

where \(A\) and \(\ell\) are the respective cross-sectional area and effective length of the tube-shaped opening, \(V\) is the internal volume of the resonator cavity, and \(c\) is the speed of sound in air (343 m/s). The effective length of the opening is somewhat longer than the actual length due to the fact that some of the air just inside and just outside the opening is also moving. A good approximation for the effective length is

\[
\ell = L + 1.45r \tag{11.5}
\]

where \(L\) and \(r\) are the actual length and radius of the neck. Further discussion as well as a derivation is available at http://www.phys.unsw.edu.au/jw/Helmholtz.html.

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1The energy required to establish a magnetic field in an inductor is \(\frac{1}{2}L^2\), which looks a lot like kinetic energy of a moving mass \(\frac{1}{2}mv^2\). The energy required to build up charge in a capacitor is \(\frac{1}{2}Q^2\), which looks a lot like potential energy in a spring \(\frac{1}{2}kx^2\).
11.3 Equipment

Magnetically-driven mass-spring oscillator system with VI control. Helmholtz oscillator (small speaker, Wavetek signal generator Christmas bulb, lump of clay, computer microphone). Digital frequency meter.
Quiz

Q11.1 In the analogy between an LRC circuit and a mass-spring oscillator, a *stiff* spring is like a
(a) large inductance.
(b) small inductance.
(c) large capacitance.
(d) small capacitance.

Q11.2 The resonance frequency of the mass/spring oscillator does *not* depend on
(a) the mass.
(b) the spring constant.
(c) the damping coefficient $b$.

Q11.3 In the analogy between an LRC circuit and an acoustical resonator, a *large* resonator volume is like a
(a) large inductance.
(b) small inductance.
(c) large capacitance.
(d) small capacitance.

Q11.4 Consider a damped mass-spring oscillator with $m = 50$ g, $k = 80$ N/m, and $b = 25$ g/s that is driven at $f = 2.5$ Hz by a vibrator with peak force amplitude $F_0 = 1$ N.
(a) What is the resonance frequency $f_0$?

(b) Determine the maximum velocity $v_0$ of the mass.

(c) Use energy conservation, $\frac{1}{2} m v_0^2 = \frac{1}{2} k x_0^2$, to determine the maximum displacement $x_0$. 

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Exercises

Half of the class should start on part A, while the other half starts on part B. Roughly halfway through the lab period, the A and B lab teams will switch places.

A: Measure the frequency response of a damped mass-spring oscillator.

L11.1 Working with a team of up to three students, familiarize yourself with the damped mass-spring resonator apparatus. Use either of two available stations. Analog output channel 1 on the interface box drives a current through the large electromagnet, which creates the magnetic force that drives the mass-spring system. Analog input channel 2 (called ai1 by Labview) reads in the voltage from the smaller coil, which is proportional to the velocity of the mass.

L11.2 Given the mass, the spring constant, and the damping coefficient of your apparatus, predict the resonance frequency of your oscillator. The combined mass \( m \) (of the rod, magnet, and damper) is printed on the front of the apparatus. The spring constant \( k \) can be approximated by resting a known mass (between 100 g and 500 g) on the damper and observing the resulting displacement. The approximate damping coefficient \( b \) should be printed on the underside of the damper disc. Enter the resulting \( b, m, \) and \( k \) values into your lab notebook. Use “impedance.nb” to plot the expected "response" curve vs frequency.

L11.3 Use the “HarmonicOscillator” VI to map out the frequency-dependent velocity of the oscillator over a reasonable frequency range that includes both the low and high-frequency tails of the resonance. Collect about 30 data points, manually recording the peak-to-peak signal amplitude (with estimated error) at each point. Choose an an appropriate linear frequency range that includes the interesting features and which approximately centers the resonance in the middle of range. Enter the data into Excel and plot it for your lab notebook. Show how the measured resonance frequency compares to your predicted value? They should be quite close.

L11.4 Use Mathematica (“NLSQ.nb”) to fit a resonance curve to your data. In contrast to the approach taken with the electrical LRC circuit, fix the mass at the printed value and refine both \( b \) and \( k \), along with the scale factor. Make sure that your result is reasonable and record it in your lab notebook along with error estimates for the fitting parameters.

B: Measure the frequency response of an acoustical Helmholtz resonator.

L11.5 Choose a Helmholtz resonator cavity from among the objects provided. Excite the resonance by blowing a gentle stream of air across the end of
the opening. With a little practice, you may be able to produce a clean musical note, though you shouldn't expect to outshine Fritz Richard after only one day.

L11.6 Mount a small acoustic driver (i.e. speaker) a short distance away from the mouth of the resonator and drive it with a sinusoidal waveform from the Wavetek signal generator. Search the frequency range from 100 Hz to 600 Hz to locate the Helmholtz resonance. When you find the resonance, the audible sound will get much louder as speaker approaches the cavity. (Don't be fooled by a higher-frequency resonance of your inexpensive speaker driver.) Once you find the resonance, adjust the distance between the driver and the resonator until you maximize the response. Use rods and clamps to fix the speaker in an optimal position.

L11.7 Measure the acoustic response by dangling a small microphone into the center of the cavity by its leads, and then plug it in to the microphone jack (pink) on the front of your computer. Create a labview VI that reads in a 0.10 second sample from the microphone at a 20 kHz sample rate, and sends the output into a Graph. Use the Express → Input → Sound Input VI, and create numeric controls to set the sample rate and the sample duration. Place these items inside of a “While” loop, and set it to run continuously. If your microphone is not correctly connected, the VI throws and error message and occasionally hangs the computer (blame Windows 7). In the frequency range below 1 kHz, the graph should show a clean sinusoidal signal coming in from your mic that gets updated several times per second.

L11.8 Map out the frequency response of the resonator over a reasonable frequency range that includes both the low and high-frequency tails of the resonance. Use a digital frequency meter to accurately measure your signal frequency. Collect about 30 data points, manually recording the peak-to-peak signal amplitude at each point. Include amplitude error estimates. Record your data in an Excel spreadsheet and plot the results on a linear horizontal scale for your lab notebook. Note that you don't need to attempt a quantitative fit to the data, though you can if you want to.

L11.9 Use the expression in the introduction to predict the resonance frequency. How does your prediction compare to the observed resonance frequency? If there is a discrepancy, can you explain it?
Lab 12

Fourier Analysis

While the uninitiated may feel more comfortable viewing waveforms (or signals) in the time domain, scientists and engineers often prefer to transform a signal into the frequency domain for further analysis. This approach, called Fourier analysis, is a powerful method of analyzing oscillatory signals, such as sound waves. In this lab, you will generate and analyze the frequency spectra of a variety of signals.

12.1 Functions Built from Sinusoidal Waves

Consider the following (somewhat arbitrary) function composed of sine waves added together:

\[ f(t) = \frac{1}{2} - \sum_{n=1}^{N} \frac{(-1)^n}{\pi n} \sin(n\omega_0 t) \]  

(12.1)

If we sum to \( N = 1 \), for example, the above function is \( f(t) = \frac{1}{2} + \frac{\sin(\omega_0 t)}{\pi} \), which is graphed in Fig. 12.1. If we sum to \( N = 3 \), the function is \( f(t) = \frac{1}{2} + \frac{\sin(\omega_0 t)}{\pi} - \frac{\sin(2\omega_0 t)}{2\pi} + \frac{\sin(3\omega_0 t)}{3\pi} \), which is graphed in Fig. 12.2. In the latter case, the waveform is more interesting.

The case of \( N = 10 \) is plotted in Fig. 12.3. As you may suspect, the coefficients \( \frac{(-1)^{n+1}}{\pi n} \) were chosen especially to produce a sawtooth wave. In the limit \( N \to \infty \), the high-frequency components join to make the sawtooth waveform perfectly sharp and exact (see Fig. 12.4). In general, sharp features in a waveform require the superposition of very high frequency waves.\(^1\)

Because all frequencies used are multiples of \( \omega_0 \), \( f(t) \) repeats with period \( T = \frac{2\pi}{\omega_0} \). We could have built a function similar to (12.1) using cosines instead of sines. In this case, the waveform would look the same except for being phase shifted in time.

By adding together many sinusoidal waves in just the right way, it is possible to make any waveform shape desired. This was the assertion made by Joseph

---

\(^1\)This is known as the uncertainty principle: a greater range of frequencies is needed to make sharp features in a waveform.
12.2 Discrete Fourier Series

Any periodic function can be represented by the sum

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in \Delta \omega t} \]  

(12.2)

This is called a Fourier series. Note that \( e^{-i \Delta \omega t} \) contains both sine and cosine terms.\(^2\) \( f(t) \) can only be a real function if \( c_{-n} = c_n^* \) for all \( n \). By inspection, we see that all terms in (12.2) repeat with a maximum period of

\[ T = \frac{2\pi}{\Delta \omega} \]  

(12.3)

The period of the function is such that \( f(t) = f(t+T) \).

Sometimes, scientists and engineers describe a periodic function \( f(t) \) simply by listing the coefficients \( c_n \) that multiply each term in (12.2). Since all of the important information is carried by the coefficients, this shorthand notation can be very insightful. The collection of such coefficients is known as the discrete spectrum of the function. Note that each coefficient is associated with a frequency \( \omega = n \Delta \omega \).

**Example 12.1**

Find the Fourier coefficients \( c_n \) of the sawtooth wave given by (12.1).

**Solution:** We may write (12.1) as

\[ f(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi n} e^{i n \omega_0 t} + e^{-i n \omega_0 t} \]

\[ = \frac{1}{2} \sum_{n=-\infty}^{\infty} -i(-1)^n e^{-i n \omega_0 t} + \frac{1}{2} \sum_{n=1}^{\infty} -i(-1)^n e^{-i n \omega_0 t} \]

Equation (12.2) is equivalent to writing the expansion

\[ f(t) = \sum_{n=0}^{\infty} a_n \cos(n \Delta \omega t) + b_n \sin(n \Delta \omega t) \]

The sines and cosines can be converted using Euler’s formulas (6.19):

\[ f(t) = \sum_{n=1}^{\infty} \frac{a_n}{2} e^{i n \Delta \omega t} + \frac{b_n}{2} e^{-i n \Delta \omega t} \]

\[ = a_0 + \sum_{n=1}^{\infty} \frac{a_n - i b_n}{2} e^{i n \Delta \omega t} + \sum_{n=1}^{\infty} \frac{a_n + i b_n}{2} e^{-i n \Delta \omega t} \]

from which we recover (12.2) if we set \( c_{n<0} \equiv \frac{a_n - i b_n}{2}, c_{n>0} \equiv \frac{a_n + i b_n}{2} \), and \( c_0 \equiv a_0 \). The real parts of the \( c_n \) are connected with cosine terms, and the imaginary parts of the \( c_n \) are connected with sine terms.

\(^2\) Equation (12.2) is equivalent to writing the expansion

\[ f(t) = \sum_{n=0}^{\infty} c_n e^{-i n \Delta \omega t} \]
With $\Delta \omega = \omega_0$, the coefficients $c_n$ are easily extracted (see Fig. 12.5).

We need to find the coefficients $c_n$ that allow us to construct a given function $f(t)$. There is a trick for doing this, which is outlined in appendix 12.A. The coefficients are found with the following formula:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{in\Delta \omega t} \, dt$$  \hspace{1cm} (12.4)

This completes the circle. If we know the function $f(t)$, we can find the coefficients $c_n$ via (12.4); and, if we know the coefficients $c_n$, we can generate the function $f(t)$ via (12.2).

**Example 12.2**

Find the Fourier coefficients necessary to produce the square wave depicted in Fig. 12.6. The function is defined by

$$f(t) = \begin{cases} 1 & |t| \leq \frac{T}{4} \\ 0 & |t| > \frac{T}{4} \end{cases}$$

in the range $-\frac{T}{2}$ to $\frac{T}{2}$.

**Solution:** The lowest frequency we require is $\Delta \omega = 2\pi/T$. According to (12.4), the coefficients are computed as follows:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{in\Delta \omega t} \, dt = \frac{1}{T} \int_{-T/4}^{T/4} e^{in\Delta \omega t} \, dt$$

$$= \frac{1}{T} \left[ \frac{e^{in\Delta \omega t}}{i n \Delta \omega} \right]_{-T/4}^{T/4} = \frac{1}{n \pi} \frac{e^{in\pi} - e^{-in\pi}}{2i} = \frac{1}{n \pi} \sin \left( \frac{n \pi}{2} \right)$$

These coefficients are graphed in Fig. 12.7. According to (12.2), our square wave may be written as

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{n \pi} \sin \left( \frac{n \pi}{2} \right) e^{-in \Delta \omega t}$$

**12.3 Time and Frequency Domains**

To summarize the results of section 12.2, we have two ways of specifying a periodic function $f(t)$ (with period $T$), either by the values of the function itself, or by the coefficients $c_n$ that can be used in a Fourier series that represents the function. It is insightful to write down (12.2) and (12.4) side by side:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{in\Delta \omega t} \, dt \quad \text{and} \quad f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in \Delta \omega t} \quad (T = \frac{2\pi}{\Delta \omega}) \hspace{1cm} (12.5)$$
But what about waveforms that do not repeat (for example, a single time pulse)? No problem. We simply take the limit \( T \to \infty \) and make the period infinitely long. This is equivalent to the limit \( \Delta \omega \to 0 \), which makes the frequencies in the series \( (\omega = n \Delta \omega) \) very finely spaced. In this limit, the summation in (12.5) turns into an integral (as outlined in appendix 12.B). Then our pair of equations in (12.5) becomes

\[
F(\omega) \equiv \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \quad \text{and} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega \quad (12.6)
\]

Here, \( F(\omega) \) is the continuous analog of the discrete coefficients \( c_n \). Either function, \( f(t) \) or \( F(\omega) \), contains complete information information about the time signal. Notice the nice symmetry between \( f(t) \) and \( F(\omega) \). When working with \( f(t) \), one is said to be in the \textit{time domain}, and when working with \( F(\omega) \), one is said to be in the \textit{frequency domain}.

The first formula \textit{transforms} the \textit{time-domain} signal \( f(t) \) into the \textit{frequency-domain} function \( F(\omega) \). This is called a \textit{Fourier transform}. The latter formula transforms \( F(\omega) \) back into \( f(t) \), and it is called an \textit{inverse Fourier transform}. We have exploited the concept of a \textit{functional transform}, which by definition, takes one function as input, and returns another function as output. Similar to the way a function \( f(t) \) takes in a value \( t \) and returns \( f \) as output, the notation \( \mathcal{F}\{f(t)\} = F(\omega) \) indicates a transform, which takes the whole function \( f(t) \) as input and returns \( F(\omega) \) as output.

**Example 12.3**

Find the spectrum of \( f(t) = Ae^{-t^2/\tau^2} \cos(\omega_0 t) \), which is shown in Fig. 12.8, where we have assigned \( \omega_0 = 10\pi/\tau \).

**Solution:** The Fourier transform of the waveform is

\[
F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/\tau^2} \cos(\omega_0 t) e^{-i\omega t} \, dt = A \int_{-\infty}^{\infty} e^{-t^2/\tau^2} (e^{i\omega_0 t} + e^{-i\omega_0 t})/2 \, e^{-i\omega t} \, dt
\]

where to make the integration easier we have installed \( \cos(\omega_0 t) = (e^{i\omega_0 t} + e^{-i\omega_0 t})/2 \). (Use \( \int_{-\infty}^{\infty} e^{-At^2 + Bt} \, dt = \sqrt{\pi/A} e^{B^2/4A} \). The Fourier transform, shown in Fig. 12.8, works out to be

\[
F(\omega) = \sqrt{\pi \tau A/2} \left[ e^{-\tau^2(\omega - \omega_0)^2/4} + e^{-\tau^2(\omega + \omega_0)^2/4} \right]
\]

For the curious, we can recover the original waveform via the inverse Fourier transform:

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega) \, d\omega = \tau A \int_{-\infty}^{\infty} e^{-t^2/\tau^2} \left[ \int_{-\infty}^{\infty} e^{-\tau^2(\omega - \omega_0)^2/4} \, d\omega \right] e^{i\omega t} \, d\omega + \int_{-\infty}^{\infty} e^{-t^2/\tau^2} \left[ \int_{-\infty}^{\infty} e^{-\tau^2(\omega + \omega_0)^2/4} \, d\omega \right] e^{i\omega t} \, d\omega
\]

\[
= Ae^{-t^2/\tau^2} \frac{\sin(\tau \omega_0 t)}{\tau\omega_0}
\]

where a change of integration variable \( \tilde{\omega} \equiv \omega \pm \omega_0 \) is handy.
If the pulse duration $\tau$ of the Gaussian envelope in example 12.3 were longer, the spectral peak would be more concentrated at $\omega_0$. A shorter duration $\tau$ would give a wider spectral peak, meaning more frequencies comprising the waveform.

12.4 Power Spectrum

More often than not, we are interested in the power spectrum of a waveform:

$$P(\omega) \equiv |F(\omega)|^2$$

This is what a graphic equalizer (see Fig. 12.10) plots, which you have probably seen before. Each vertical column of the equalizer represents a narrow frequency range, and shows a bar height that indicates the contribution to the waveform from that frequency range. For real waveforms $f(t)$, such as in example 12.3, the Fourier transform at positive frequencies is redundant with that at negative frequencies. It is therefore common just to look at the positive side of the graph.

In the case of a sound wave, the intensity of the sound at each frequency is proportional to $|F(\omega)|^2$. Moreover, our ears are not very sensitive to the relative phases between different frequency components that make up a sound. The bar height in Fig. 12.10 actually represents $|F(\omega)|^2$. The power spectrum, which is displayed by an equalizer, is a good representation of what we hear. Because the phase information is not present in the power spectrum, the function $f(t)$ cannot be recovered from it. In spite of the lacking information, the power spectrum is intuitive and contains arguably the most important information about the time signal.

12.5 Numerical Fourier Transforms

A computer approximates the integrals like those in (12.6) as a series or sum. For numerics, it is natural for a computer to use a Fourier series, where even the integral in (12.5) is represented by a series. This is handy for experimental data, since a measured waveform is typically sampled at even time steps. It is important that $f(t)$ be sampled sufficiently often so that it is well represented by the discrete points. The Nyquist criterion requires that the sampling rate should be at least twice the highest frequency that you care about in your Fourier series.

Experimentally, it is only practical to sample a waveform for a finite duration. Whatever the waveform $f(t)$ that is sampled, the computer will assume that the signal continues to repeats outside of the sampled range. For computation purposes, the computer takes the period $T$ to be the sampled range. If the waveform repeats with a shorter period, that’s okay; in fact, that’s good. The computer

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3 There is an optimized algorithm called a fast Fourier transform or FFT for doing this, which requires the number of data points representing the function to be $2^q$, where $q$ is an integer. In Labview uses the FFT; if you have a different number of data points, Labview will pad it with extra zeros until the number of data points is an integer power of 2.
doesn’t know what the actual period of the waveform is, and often times, neither do you. You might only find out the period after you see the spectrum. Generally, it is good practice to sample the data over a time window $T$ that is much longer than the period of a repeating waveform. By so doing, you increase your certainty about the periodic nature of the signal.

If $N$ is the number of data points and $T$ is duration of the computer’s sampling time window, then the time interval between sampled data points is

$$\Delta t = T/N$$

(12.7)

The frequency interval (or the computer’s resolution) is

$$\Delta \omega = \frac{2\pi}{T}$$

(12.8)

The computer typically computes the same number of coefficients $c_n$ as there are original sampled data points $N$. Therefore, the frequency range is

$$\Omega = \Delta \omega \cdot N = \frac{2\pi N}{T}$$

(12.9)

While we often label the time range from 0 to $T$ (although $-T/2$ to $T/2$ is just as good), the frequencies range is $-\Omega/2, \cdots -2\Delta \omega, -\Delta \omega, 0, \Delta \omega, 2\omega, \cdots, \Omega/2$. These enumerate the horizontal axis, while the $c_n$ are plotted along the vertical axis.

A discrete Fourier transform when carried out for a very long sampling times $T$ starts to look like the continuous transforms (12.6). Periodic time signals, when represented with a continuous transform $F(\omega)$, have discrete frequency spectra. If the computer’s time window $T$ is much longer than the waveform period $2\pi/\Delta \omega$, a series of sharp spikes will dominate the spectrum (at integer multiples of the function’s fundamental frequency $f_0 = \omega_0/2\pi$. The power spectrum in this case will look similar to Fig. 12.14).

A pure musical note, regardless of the voice or instrument, is a periodic time signal that the signal repeats itself in time every so often. For a truly sinusoidal signal with frequency $f_0$, the power spectrum only contains one peak at $f_0$. But a more interesting periodic function may contain many harmonics (e.g. $f_0, 2f_0, 3f_0, 4f_0, 5f_0, \ldots$).

A waveform doesn’t necessarily have to be short in duration to have wide spectrum. A continuous random waveform (called) white noise can have a very wide spectrum. When a waveform encounters a physical system, the spectrum is often modified by the frequency response function of the system.

### 12.6 Equipment

Computer equipped with microphone and headphones, tuning fork set, balloons to pop.

---

4Technically, the upper frequency should be $\Delta \omega (N/2 − 1)$ for there to be $N$ points.
Appendix 12.A  Derivation of Fourier Coefficients

To derive (12.4), we multiply both sides of (12.2) by \( e^{im\Delta \omega t} \), where \( m \) is an integer, and integrate over the period \( T = 2\pi / \Delta \omega \):

\[
\int_{-T/2}^{T/2} f(t) e^{im\Delta \omega t} dt = \sum_{n=-\infty}^{\infty} c_n \int_{-T/2}^{T/2} e^{i(m-n)\Delta \omega t} dt = \sum_{n=-\infty}^{\infty} c_n \frac{e^{i(m-n)\Delta \omega t}}{i(m-n)\Delta \omega} T/2 \]

The factor \( \sin [(m-n)\pi] / [(m-n)\pi] \) is zero if \( n = m \). If \( m = n \), then

\[
\int_{-\pi/\Delta \omega}^{\pi/\Delta \omega} e^{i(m-n)\Delta \omega t} dt = \int_{-\pi/\Delta \omega}^{\pi/\Delta \omega} dt = \frac{2\pi}{\Delta \omega}
\]

Only one term in the series survives and we get

\[
\int_{-T/2}^{T/2} f(t) e^{im\Delta \omega t} dt = T c_m
\]

Since \( m \) is a dummy index, we are free to change it to \( n \).

Appendix 12.B  Continuous Fourier Transforms

Here we derive (eq:12.9). We combine the two formula (12.5) and also take the limit \( \Delta \omega \to 0 \), which gives

\[
f(t) = \frac{1}{2\pi} \lim_{\Delta \omega \to 0} \sum_{n=-\infty}^{\infty} \left[ e^{-in\Delta \omega t} \int_{-\infty}^{\infty} f(t') e^{in\Delta \omega t'} dt' \right] \Delta \omega
\]

We start with the function \( f(t) \) followed by a lot of computation and obtain the function back again! This is not as pointless as it may seem, for in (12.8) we can recognize the definition of an integral.\(^5\)

\[
\int_{-\infty}^{\infty} g(\omega) d\omega = \lim_{\Delta \omega \to 0} \sum_{n=-\infty}^{\infty} g(n \Delta \omega) \Delta \omega
\]

\(^5\) A short calculus review may be helpful. Recall that an integral is really a summation of rectangles under a curve with finely spaced steps:

\[
\int_{a}^{b} g(\omega) d\omega = \lim_{\Delta \omega \to 0} \sum_{n=0}^{b-a} g(a + n \Delta \omega) \Delta \omega = \lim_{\Delta \omega \to 0} \sum_{n=0}^{b-a} \frac{b-a}{\Delta \omega} \left( a + \frac{b-a}{2} + n \Delta \omega \right) \Delta \omega
\]

The final expression has been manipulated so that the index ranges through both negative and positive numbers. If we next set \( a = -b \) and take the limit \( b \to \infty \), we obtain (12.9).
In our case, \( g(n \Delta \omega) \) represents everything inside the square brackets of (12.8).

The result is the Fourier integral theorem:

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \left[ \int_{-\infty}^{\infty} f(t') e^{i\omega t'} dt' \right] d\omega \tag{12.12}
\]

The piece in brackets is called the *Fourier transform*, and the rest of the operation is called the *inverse Fourier transform*.\(^6\) These may be written separately as in (12.6).

\(^6\)It is arbitrary which is called the transform and which is called the inverse transform. Also, the factor \( 1/2\pi \) may be placed on either the transform or the inverse transform, or \( 1/\sqrt{2\pi} \) can be applied to both.
Quiz

Q12.1 The Nyquist sampling criterion dictates that a periodic signal should be sampled
(a) exactly twice per period.
(b) at least twice per period.
(c) no more than twice per period.

Q12.2 The power spectrum of a signal is the
(a) absolute magnitude of the Fourier transform.
(b) square of the absolute magnitude of the Fourier transform.
(c) square root of the Fourier transform.
(d) phase of the Fourier transform.

Q12.3 In Fig. 12.15 showing input and output, which time-domain signal has the widest frequency spectrum?
(a) The solid square wave.
(b) The dashed-line wave.

Q12.4 For a time signal that is sampled at 5 kHz for 0.2 s, compute the following:
(a) the number of data points \( N \).

(b) the temporal range.

(c) the temporal resolution (i.e. spacing between time points).

(d) the frequency range.

(e) the frequency resolution (i.e. spacing between frequency points).
Exercises

A. Become familiar with the Labview VI for generating Fourier spectra.

L12.1 Copy the “Fourier_waveform.vi” to your own working area. Open it up and explore its front panel and block diagram. Use the VI to explore the Fourier power spectra of several computer-generated waveforms. Develop some intuition as to how the variable waveform parameters affect the spectra. Include and describe a few interesting results in your lab notebook.

Learn how to switch between linear and logarithmic scales on the X and Y axes. Ask your TA to show you how to use the Graph Palette to zoom in on features of interest.

B. Generate and analyze Fourier spectra of sound-wave samples.

L12.2 Copy the “Fourier_sound.vi” to your own working area. Open it up and explore its front panel and block diagram. Turn off the Enable Filter switch and set the Card/Interface switch to “Sound Card” on the VI front panel. Set the sample rate at 20 kHz and the duration to 1 s. Plug the headset microphone into the microphone jack (pink) at the back of your computer, and the headset speaker in the speaker jack (green). Collect a few samples of yourself whistling a steady note, and observe the relevant peaks in the resulting power spectrum. You should be able to hear the playback of your sound sample in the headphones. Ignore the filtered time and frequency-domain graphs at the right-hand side of the front panel.

L12.3 Now record the note produced by one of the tuning forks provided and generate its Fourier power spectrum. Use the spectrum to accurately determine the fundamental vibration frequency of the fork (a log scale or zoom view may help). Try two more forks. How well do the measured frequencies agree with the nominal values printed on the tuning forks themselves? Print a few time and frequency graphs for your lab notebook along with your observations and explanation.

L12.4 The sample rate that you choose is dictated in part by the maximum frequency that you want to observe in your spectrum - the sample rate must be twice the maximum frequency. This is called the Nyquist sampling criterion. For example, if you want to measure frequencies up to 10 kHz, you need to collect at least 20,000 samples/second. To see this, choose a tuning fork with a fundamental frequency $f_0$ somewhere near 1 kHz. For convenience, use the graph palette to zoom in on a time region containing about 10 fundamental oscillations and turn the x-axis autoscaling off. Start the sample rate at a value equal to roughly $4f_0$, using a time duration of 1 second. Then lower the sample...
frequency to $3f_0$, and then to $2f_0$, and finally to $f_0$. What happens to your time and frequency-domain signals when the Nyquist frequency (i.e. half the sample rate) drops below the frequency of the sound that you are measuring? Print a few informative graphs for your lab notebook and explain what you see there.

**L12.5** Generate a Fourier power spectrum for a few abrupt sounds such as a clapping, yelling, or popping a balloon. Adjust the sample frequency and sample time if necessary. Once again, record your results and observations in your lab notebook.

**C. Use Fourier voice spectra for vowel-sound recognition.**

**L12.6** Generate Fourier spectra for each of the following vowel sounds: â, ê, ô, ū (pronounce as "ooh" not "you"). You may need practice a few times before obtaining good spectra. Note that the cleanest spectra generally come from notes that are sung rather than spoken. It is also important to hold the note steady during the entire sampling interval.

**L12.7** Print each spectrum and label it lightly on the back, but not on the front. Each lab partner should generate their own spectra. Now see if you can identify the vowels associated with each of your lab partner's (or someone else's) spectral printouts without looking at the labels on the back. Then save your spectra in your lab notebook along with your observations. Discuss which harmonics tend to be emphasized for each vowel sound.

**D. Apply frequency filters to vocal input.**

**L12.8** Increase your sampling interval to 2 or more seconds (at a 20 kHz sampling rate). Record vocal input with the microphone, and verify that you can hear the playback clearly in the headphones. On the front panel, turn the “Filter Enable” switch to “On”. You should now see (and hear) that a Butterworth band-pass filter has been applied to your input signals. The unfiltered signals appear in the graphs on the left-hand side of the front panel, while the filtered signals appear on the right-hand side.

**L12.9** Try a variety of filter cutoff settings and observe the effects of the filter on both time and frequency-domain signals. Print graphs that illustrate a couple of interesting cases for your lab notebook, and include observations and explanation.
Lab 13

Loudspeakers

In this lab, you will bring together all of your experience with data-analysis techniques, Fourier spectral measurements, automated experiment-control and data-acquisition methods, and AC impedances to solve an interesting practical problem: design and build a crossover network to drive a loudspeaker.

13.1 Speaker Drivers

A loudspeaker is usually described as a multi-component system that includes an enclosure, two or more speaker drivers and an electrical crossover network. Have you ever wondered what physically differentiates a high-quality loudspeaker from a low-quality loudspeaker?

In the context of acoustics, a transducer is a device that converts mechanical (or acoustical) energy into an electrical current, whereas a generator does the opposite. In this context, a microphone is a transducer and a speaker driver is a generator. Both have physical responses that are time and frequency dependent. The time and frequency-response functions are related to one another via the Fourier transform. If the frequency-response function (FRF) were perfectly flat (i.e. frequency independent), then the output signal would always match the input signal – a very desirable outcome. Inevitably, however, real transducers and generators perform better at some frequencies than at others. A woofer driver, for example, is designed to respond strongly to low-frequency audio signals, but does not respond well to high-frequency signals. A tweeter, on the other hand, is designed to respond strongly to high-frequency audio signals, but not low-frequency signals.

The generic speaker driver in Fig. 13.2 is much like a damped driven mass-spring oscillator. The relatively solid “cone” is the moving mass. The “spider” and the “surround” are the springs, but also provide some damping. A small “voice-coil” electromagnet attached to the cone interacts with a stationary permanent magnet to produce a time-varying driving force. When an audio signal is applied to the voice coil, it excites a corresponding mechanical motion of the cone, which then generates pressure waves in the air outside the cone, which propagate.
outward as sound.

13.2 Loudspeakers

Ideally, we are looking for a flat FRF that leaves the input signal undistorted. It would be nice to have a single speaker driver that has a uniform FRF throughout the audio range (0 to 20 kHz). In practice real speaker drivers just aren't that good. A typical loudspeaker uses two or more drivers in parallel (see Fig. 13.3) with complimentary frequency response functions to cover different regions of the audio frequency range. Typical FRFs for woofer and tweeter speaker drivers are shown in Figs. 13.4 and 13.5. Observe that the woofer response in Fig. 13.4 is effectively that of a low-pass filter with a cutoff near 2 kHz, whereas the tweeter response in Fig. 13.5 is effectively that of a high-pass filter with a cutoff near 1.5 kHz.

![Figure 13.3](image1.png)
Figure 13.3 A woofer (bottom) and tweeter (top) connected in parallel using a very basic crossover network.

![Figure 13.4](image2.png)
Figure 13.4 Frequency response (Hz) of a woofer measured in our lab.

![Figure 13.5](image3.png)
Figure 13.5 Frequency response (Hz) of a tweeter measured in our lab.

The idea is to combine these two FRFs so that a wider range of frequencies can experience uniformly strong gain (see 13.6). As long as the individual ranges of the drivers have some overlap (to avoid frequency gaps), the entire audio range can be covered. Unfortunately, real drivers have mechanical and acoustical resonances that further complicate the gain curves. A more expensive driver will usually be flatter over a wider range. The FRFs of the various drivers also need to be balanced because we don't want the tweeter to have a much larger response at 10 kHz than the woofer has at 500 Hz.
13.3 Crossover Network: Concept

We want to selectively send to each driver only the frequencies in its designed range. This is accomplished with an electrical crossover network, which splits the input signal into several different outputs, each of which is filtered to match the right frequency range of a given driver. A crossover network can be as simple as the inline capacitor and inductor shown in Fig. 13.3.

To better appreciate the crossover network, consider the two circuits shown in Fig. 13.3, where we take (for simplicity sake) the impedance of each speaker to be purely resistive with values $R_W$ and $R_T$. The inductor inserted in the line to the woofer then creates an LR circuit. In lab 7 you studied the frequency response of a low-pass LR filter (see Fig. 13.7). The gain is given by

$$|G_W| = \left| \frac{R_W}{Z} \right| = \frac{R_W}{\sqrt{R_W^2 + (\omega L)^2}} = \frac{1}{\sqrt{1 + \omega^2 (L/R_W)^2}} \quad (13.1)$$

If you choose the cutoff wisely, this electrical circuit will pass only the low frequency range that matches the mechanical/acoustic response of the woofer so that the woofer never receives frequencies that it can't handle.

The capacitor inserted in the line to the tweeter creates an RC circuit. In lab 8, you studied the frequency response of a high-pass RC filter (see Fig. 13.8). The gain is given by

$$|G_T| = \left| \frac{R_T}{Z} \right| = \frac{R_T}{\sqrt{R_T^2 + (1/\omega C)^2}} = \frac{\omega R_T C}{\sqrt{1 + \omega^2 (R_T C)^2}} \quad (13.2)$$

Again, you can choose the cutoff frequency of this electrical circuit to pass only the high frequencies that match the mechanical/acoustic response of the tweeter. When the high and low frequency cutoffs of the two crossover channels are made equal (i.e. $L/R_W = R_T C$), the rules of parallel-circuit addition (see section 9.4) leads to an exactly flat gain curve, as seen in Fig. 13.9:

$$|G_{net}| = \sqrt{|G_W|^2 + |G_T|^2} = 1 \quad (13.3)$$
What could it hurt to just send the total input signal to both drivers, since they won't respond well to frequencies outside their respective ranges anyway? There are at least two important reasons to use a crossover network. 1) In a frequency-overlap region, where two adjacent drivers both produce significant output, the two drivers become acoustically coupled via the pressure waves that they generate. In this way, one driver can modify (i.e. degrade) the FRF of another driver. 2) The waves produced by the two drivers can interfere to create strange spatial distributions in the sound intensity.

13.4 Crossover Network: Implementation

In this lab, you will build a simple first-order two-way (i.e. two driver) loudspeaker system with one woofer and one tweeter. After splitting the signal into two branches, you will want filter the woofer signal with a low-pass RL filter and the tweeter signal with a high-pass RC filter. By setting the cutoff frequencies of the two filters equal to one another, the FRF of the overall loudspeaker system can be optimized to be as flat as possible. In Fig. 13.9, the separate responses of a low-pass RL filter and a high-pass RC filter have been tuned to a mutual cutoff frequency. More complicated filter circuits with sharper cutoffs (employed by most commercial loudspeakers) can reduce the amount woofer/tweeter overlap.

The enclosure is also an important component of the loudspeaker. It is typically designed to extend the low-frequency end of the response curve of the woofer. No woofer has a low-frequency response that stays flat all the way to zero frequency – they typically drop off somewhere between 100 and 200 Hz. The enclosure adds a mechanical potential-energy (i.e. analogous to a spring or a capacitance) to the system since the enclosed volume of air is compressed by a pressure wave. Combined with the mass of the driver cone, you get a mass-spring oscillator that can be tuned to resonate broadly just below the woofer's low-frequency cutoff, effectively extending the low-frequency response of the system.

Take a moment to read the hyperphysics entry on crossover networks. This page is located at http://hyperphysics.phy-astr.gsu.edu/hbase/audio/cross.html. Notice that the electrical resistance of the driver itself can be considered to be a filter component. Ideally, one treats each driver in Fig. 13.3 as a simple electrical resistor (e.g. 8 Ohm) in order to obtain the desired high- and low-pass filter FRFs. But in reality, the electrical, mechanical and acoustic dynamics of a speaker driver are coupled, resulting in an effective electrical impedance that acts like a more complicated LRC network. The network in Fig. 13.3 is common but very unsophisticated.

The above hyperphysics site also shows you how to use a parallel resistance to reduce the output from the tweeter. This "balance" resistor allows you to reduce the sound intensity of the tweeter to match that of the woofer. In this lab, we will compensate for any imbalance by simply moving the mic closer to one driver than the other. But at least you know how to apply a balance resistance if you need to.
13.5 Measuring the Frequency Response Function

There are several approaches to measuring the FRF of a loudspeaker. We will mention three here.

1) As in previous labs in which you measured the FRFs of electrical, mechanical and acoustical systems, you can manually sweep the frequency of a sinusoidal input signal over a wide range of frequencies to see the gain at a number of points along the way. This is called the continuous wave (CW) approach.

2) You can send in a very sharp input pulse, and measure the acoustic time response with a microphone, which can then be Fourier transformed to find the FRF of the driver. In lab 12, you found that a pulsed signal (e.g. a hand clap or a balloon popping) produces a very broad frequency spectrum (with synchronized phase). An infinitely-sharp pulse, if it can be produced, provides a perfectly flat input frequency spectrum.

![Figure 13.10 White noise.](image)

3) You can send random noise (or static) into the loudspeaker (see Fig. 13.10). Such a signal is often called white noise because it contains a uniform mixture of frequencies with random phase, much like incoherent sunlight, which has a quasi-uniform (i.e. white) spectrum over the visible range. This is perhaps the easiest and most commonly used approach for testing acoustical equipment. You will use this approach to characterize your drivers. One simply sends an artificially-generated white-noise electrical signal into the driver, and measures the frequencies picked up by a microphone.
One important caveat: we have ignored the fact that the other components in the experiment (the power amplifier, the microphone, the preamplifier, the output and input electronics, not to mention the echos in the room) also have their own FRFs. This situation is illustrated in Fig. 13.11, where the output of each component in the train becomes the input for the next component. The combined FRF is the multiplicative product of the individual FRFs of each of these components.

\[
\left( \frac{\text{output}}{\text{input}} \right)_{\text{train}} = \left( \frac{\text{output}}{\text{input}} \right)_1 \left( \frac{\text{output}}{\text{input}} \right)_2 \left( \frac{\text{output}}{\text{input}} \right)_3 \ldots \left( \frac{\text{output}}{\text{input}} \right)_n \quad (13.4)
\]

Because any component with a flat response can be harmlessly ignored (multiplying by a constant doesn't change the shape of the FRF), one might hope to assume that the auxiliary components all have flat frequency responses. But in reality, one needs to check auxiliary components very carefully to make sure that they don't have severe frequency-response limitations. A multimeter, for example, loses sensitivity at frequencies above a few hundred Hz. A good oscilloscope, on the other hand, has a flat response from 0 to well beyond 2 MHz. Thus an oscilloscope can be used to directly determine the FRFs of the signal generator and amplifiers.

For simplicity, we will assume that all components except our speaker driver have flat responses that can be neglected. Our amplifier responses are flat enough to ignore. However, the responses of the microphone and the echos in the room are not. Thus, even your flattest FRF won't look very ideal. A really expensive mic would help, as would doing the experiment in an anechoic chamber (i.e. no echo). Moving the mic very close to the drivers helps to minimize the echos from the room, though this trick is rather artificial.

### 13.6 Equipment

Computer and interface box, dynamic microphone, woofer and tweeter drivers, variable-volume loudspeaker enclosure, preamplifier, power amplifier, frequency-response VI, crossover network components.

---

1 A *caveat* is fancy word commonly used in science that means a qualification or warning.
**WARNING:** The speaker drivers are quite expensive and easily damaged.

1) The diaphragms can be easily ripped or punctured. Touch them only lightly.
2) The speaker enclosure is heavy and should be lifted and moved with care. Do not drop it.
3) The quickest way to destroy your woofer driver is to change the enclosure volume without first breaking the seal on the adjustable back plate, thereby creating a large positive or negative pressure inside. First set the plate position with the plate laying face down, and then rotate it up to make and lock the seal.
4) The quickest way to destroy your tweeter driver is to feed it too strong a signal. A signal level that sounds great from the woofer will fry your tweeter. After changing the cables on any driver, always start at zero gain on the power amplifier and turn the gain up slowly until the sound level just begins to annoy your neighbors.
Quiz

Q13.1 The purpose of a two-way crossover network is to
(a) boost the low-frequency response of the woofer.
(b) enhance the high-frequency response of the tweeter.
(c) split the input spectrum into low and high frequency components
   for separate delivery to the woofer and tweeter.
(d) balance the overall volumes of woofer and tweeter.

Q13.2 In addition to the frequency responses of your speaker drivers, your
measurements will inevitably detect the frequency responses of the
other components in your signal train. Circle the two components with
the largest non-uniform frequency responses.
(a) white noise source (analog output port).
(b) power amplifier.
(c) the room.
(d) microphone.
(e) signal preamplifier.
(f) analog input port.

Q13.3 Which of the following actions will NOT damage your loudspeaker
drivers?
(a) Failing to turn the power-amp gain knob to zero before and/or after
   changing the wiring to a driver.
(b) Moving the back wall of the enclosure without first breaking the
   pressure seal.
(c) Sending too strong a signal to the tweeter.
(d) Jamming a pencil through the woofer cone.
(e) Playing great 80's music in stereo.

Q13.4 If your 8-Ω woofer and tweeter both have cutoff frequencies near 1200
Hz, compute the optimal values of $L$ and $C$ needed for a first-order
Butterworth filter.
Exercises

A. Isolate the acoustical frequency responses of a pair of speaker drivers.

Form teams of no more than three students per station.

L13.1 Equipment configuration: The computer sends a white-noise signal out to your blue power amplifier from analog output $A$ of your interface box (ao0), and then on to your speaker driver. Your microphone picks up the signal and feeds it through your preamplifier and into analog input 2 (ai1) of the interface box, where it is Fourier transformed and the frequency response displayed. A third BNC cable tees off from output $A$ and runs into input 1 (ai0) of the interface box as a reference signal. On the microphone preamplifier, set the coarse gain to 20 and the fine gain all the way up. On the power amplifier, use the line input and set the gain knob ALL THE WAY DOWN.

L13.2 Labview configuration: Copy FrequencyResponse.vi into your own workspace. Explore the front panel and block diagram of the VI. The Getting Started panel shows you how to hook up the external cables. The FRF panel shows the magnitude of the computed frequency response function (i.e. the Fourier transform). The DAQ settings tab should be preset with correct default values so that you don’t need to modify them (though you will need to change the input ports if ao0 or ao1 don’t work on your interface box). In the FRF settings tab, set the number of samples to 10,000 (i.e. 0.5 seconds) with a Hanning-type apodization window. Set the averaging controls to 30 measurements, RMS averaging, exponential weighing, and one-shot repetition. Try a few test measurements – start the VI and turn the power amplifier gain up slowly until the speaker volume just begins to annoy your neighbors.

L13.3 Isolate the acoustic frequency response of your woofer driver over the range from 0 to 10 kHz (loudspeaker enclosure should be completely open in back). Place the mic up close to the driver (2 or 3 inches). After having your TA check to make sure that your frequency response looks reasonable, plot it (cut and paste from VI front panel into MS Word) and describe it in your lab notebook. You can blink the room lights to signal others in the room that you are taking data.

L13.4 Carefully install and seal the back plate of your loudspeaker enclosure. Now repeat the FRF measurement and plot the new FRF. Describe how the enclosure modifies the low-frequency end of the FRF. Try several different positions of the back plate, and describe the effect of enclosure volume on the low-frequency end of the FRF.

L13.5 Measure the acoustic frequency response of your tweeter driver over the range from 0 to 10 kHz. Because the tweeter is self-enclosed, it
doesn't matter whether or not the large enclosure is sealed. Plot and describe the tweeter FRF. How does it differ from the woofer FRF?

**B: Design and build a loudspeaker that includes your drivers, an enclosure, and a 2-way crossover network.**

**L13.6** Design a first-order two-way Butterworth crossover network that optimally combines your woofer and tweeter drivers to obtain the flattest possible frequency response function. The [http://www.lalena.com/Audio/Calculator/XOver/](http://www.lalena.com/Audio/Calculator/XOver/) site may be helpful; but repeat the final calculations by hand in your lab notebook. Review your woofer and tweeter FRFs, and do your best to pick a crossover frequency near the middle of the region where their FRFs overlap. Then set the 3dB rollover points of the high and low-frequency channels of your crossover equal to the crossover frequency. You should consider which audio resistors, capacitors, and inductors are available in the lab as you develop your design. Use an RLC multimeter to measure the value of each component that you use, including the resistances of your 8-Ω drivers. The actual and printed values may differ.

**L13.7** Build the crossover network that you designed (use a breadboard for any resistors or capacitors that you select). Use *FrequencyResponse.vi* to visually check the frequency-responses of the woofer and tweeter output channels to make sure that they work as expected (not for the notebook). Now drive both output channels simultaneously. Adjust the microphone position to balance the sound intensity from the two drivers. Try to obtain the flattest possible frequency response over the widest possible range. Use the optimal enclosure volume that you identified earlier. You may want to try a smaller capacitance if you get a lot of distortion in the frequency-overlap region. Plot your best FRF for your lab notebook. Consider the imperfections in your FRF between 10 Hz and 10 kHz, and describe further improvements (a better mic, a more sophisticated 3rd-order crossover network, an echo-free room, etc.) that would improve your FRF.

**L13.8** The stereo-BNC adaptor cable will allow you to connect an ipod to the power-amplifier input. Have fun, but try to accomodate those who are still collecting data. *Please do not put any music on the lab computers.* If you find that your tweeter is a little too dominant (due to the fact that we didn't require you to properly balance your drivers), suppress the tweeter intensity by placing an appropriate resistor in parallel with the tweeter driver and modifying the associated capacitance accordingly. For fun, you might want to try sending stereo-right signal to one loudspeaker and the stereo-left signal to another team's loudspeaker.
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# Physical Constants

<table>
<thead>
<tr>
<th>Constant</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permittivity</td>
<td>$\epsilon_0$</td>
<td>$8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$</td>
</tr>
<tr>
<td>Permeability</td>
<td>$\mu_0$</td>
<td>$4\pi \times 10^{-7} \text{ T} \cdot \text{m} / \text{A}$ (same as kg $\cdot$ m/C$^2$)</td>
</tr>
<tr>
<td>Speed of light in vacuum</td>
<td>$c$</td>
<td>$2.9979 \times 10^8 \text{ m/s}$</td>
</tr>
<tr>
<td>Charge of an electron</td>
<td>$q_e$</td>
<td>$1.602 \times 10^{-19} \text{ C}$</td>
</tr>
<tr>
<td>Mass of an electron</td>
<td>$m_e$</td>
<td>$9.108 \times 10^{-31} \text{ kg}$</td>
</tr>
<tr>
<td>Boltzmann's constant</td>
<td>$k_B$</td>
<td>$1.380 \times 10^{-23} \text{ J/K}$</td>
</tr>
<tr>
<td>Planck's constant</td>
<td>$h$</td>
<td>$6.626 \times 10^{-34} \text{ J} \cdot \text{s}$</td>
</tr>
<tr>
<td>Planck's constant</td>
<td>$\hbar$</td>
<td>$1.054 \times 10^{-34} \text{ J} \cdot \text{s}$</td>
</tr>
<tr>
<td>Stefan-Boltzmann constant</td>
<td>$\sigma$</td>
<td>$5.670 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4$</td>
</tr>
</tbody>
</table>