Physics 452: Homework #19

Due Tuesday, Mar. 31, 5:00PM, 2009

1. Given the Gaussian momentum distribution $a(\bar{p}) = \xi \exp\left\{-\frac{|\bar{p}-\bar{p}_o|^2}{2p_w^2}\right\}$, determine the factor ξ that normalizes the wave packet such that $\langle \Psi(\bar{r},t) | \Psi(\bar{r},t) \rangle = 1$.

2. (a) For the generic superposition wave packet $\Psi(\bar{r},t) \equiv \int a(\bar{p}) \Psi_{\bar{p}}(\bar{r},t) d^3 p$, show that $\langle \Psi(\bar{r},t) | \hat{\bar{p}} \Psi(\bar{r},t) \rangle = \int d^3 p \bar{p} |a(\bar{p})|^2$, where $\hat{\bar{p}} \equiv -i\hbar \nabla$.
(b) For the specific momentum distribution in Prob. 1, compute $\langle \Psi(\bar{r},t) | \hat{\bar{p}} \Psi(\bar{r},t) \rangle$.
HINT: In the exponent use $|\bar{p}-\bar{p}_o|^2 = (p_x - p_{ox})^2 + (p_y - p_{oy})^2 + (p_z - p_{oz})^2$. In front of the exponent use $\bar{p} = (p_x - p_{ox})\hat{x} + (p_y - p_{oy})\hat{y} + (p_z - p_{oz})\hat{z} + \bar{p}_o$ and look for odd functions integrated over even limits to kill.

3. (worth double) (a) Consider the low-energy regime where we may write $E(p) = mc^2\sqrt{1+(p/mc)^2} \cong mc^2\left[1+\left(p_x^2+p_y^2+p_z^2\right)/\left(2m^2c^2\right)\right]$ (or simply $E(p) \cong mc^2$ in the denominator of the integral). Show that the wave function for the distribution in Prob. 1 can be written as

$$\Psi(\bar{r},t) = \left(\frac{\sqrt{\pi}}{2\pi\hbar p_w \alpha}\right)^{3/2} e^{-\frac{p_o^2}{2p_w^2} - i\frac{mc^2}{\hbar}t + \frac{\beta_x^2+\beta_y^2+\beta_z^2}{4\alpha}}$$

, where $\alpha \equiv \frac{1}{2p_w^2} + i\frac{t}{2m\hbar}$, $\beta_x \equiv \frac{p_{ox}}{p_w^2} + ix$, $\beta_y \equiv \frac{p_{oy}}{p_w^2} + iy$, and $\beta_z \equiv \frac{p_{oy}}{p_w^2} + iz$. Note: $\int_{-\infty}^{\infty} du e^{-\alpha u^2 + \beta u} = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}}$
Although this expression is messy, there are no integrals left to perform! A computer can easily plot a movie of this (or of $\rho(\bar{r},t)$). Unfortunately, it is not very relativistic. In fact, it is the same result that the Schrödinger would give.
(b) Provide a condition of applicability relating to t, p_o, and p_w for this expression.
HINT: Make sure that the first neglected term in the expansion, namely $\frac{mc^2}{8}\left(\frac{p}{mc}\right)^4$, when multiplying $t/\hbar$ is much less than one. The largest momentum that contributes to the integral is on the order of $p \sim p_o + p_w$, given the Gaussian factor.