11.1 BIG HINT: To solve for the trajectory in part (a), use coordinates $r$ and $\phi$.

Note: $\theta = \pi - \phi_{\text{final}}$.

Energy $E = \frac{1}{2} m v_{\text{initial}}^2$ and an angular momentum are conserved: $L = m v_{\text{initial}} b = b \sqrt{2mE}$.

To solve for the trajectory, use

$$E = T + V \Rightarrow E = \frac{1}{2} m \left( r^2 + r^2 \dot{\phi}^2 \right) + V(r) \quad \text{Note:} \quad \ddot{r} = \frac{d}{dt} \frac{dr}{d\phi} = \frac{\dot{\phi}}{r} \frac{d}{dt} r \quad \text{and} \quad L = mr^2 \dot{\phi} \Rightarrow \dot{\phi} = \frac{L}{mr^2}.$$  

$$\Rightarrow E = \frac{1}{2} m \left( \frac{L}{mr^2} \right)^2 \left[ \left( \frac{d}{d\phi} \right)^2 + r^2 \right] + V(r),$$

from which obtain

$$\frac{dr}{d\phi} = \pm \frac{r}{b} \sqrt{r^2 - \frac{q_1 q_2}{4 \varepsilon_0 E} r}.$$  

As the particle is approaching, we have $\frac{dr}{d\phi} < 0$, and as it leaves, we have $\frac{dr}{d\phi} > 0$. The sign switches when $q_1$ is closest to $q_2$. Therefore, $\left. \frac{dr}{d\phi} \right|_{r=r_{\text{min}}} = 0$ or

$$r_{\text{min}} = \frac{1}{b} \sqrt{\frac{2 q_1 q_2}{4 \varepsilon_0 E}} \left( r_{\text{min}} - b \right) = 0.$$  

The only sensible root is

$$r_{\text{min}} = \frac{-\beta + \sqrt{\beta^2 + 4b^2}}{2} > 0,$$  

where $\beta = -\frac{q_1 q_2}{4 \varepsilon_0 E}$. Now solving for the final trajectory,

$$\phi_{\text{final}} = \int_{r_{\text{min}}}^{r_{\text{exp}}} \frac{dr}{ \left( \frac{r}{b} \sqrt{r^2 - \frac{q_1 q_2}{4 \varepsilon_0 E} r} \right)^2} + \int_{r_{\text{exp}}}^{r_{\text{min}}} \frac{dr}{ \left( \frac{r}{b} \sqrt{r^2 - \frac{q_1 q_2}{4 \varepsilon_0 E} r} \right)^2} = 2b \int_{r_{\text{min}}}^{r_{\text{exp}}} \frac{dr}{ \left( \frac{r}{b} \sqrt{r^2 - \frac{q_1 q_2}{4 \varepsilon_0 E} r} \right)^2}.$$  

Useful:

$$\int \frac{dx}{x \sqrt{\alpha x^2 + \beta x + \gamma}} = \frac{1}{\sqrt{-\gamma}} \sin^{-1} \left( \frac{\beta x + 2\gamma}{|\gamma| \sqrt{\beta^2 - 4 \alpha \gamma}} \right).$$

When the integrated expression is evaluated at $r_{\text{min}}$, you should find that the argument of $\sin^{-1}$ becomes one.

See problems on next page.

11.2
11.8 HINT: Eq. 1.102 in your E&M text (Physics 441-442): \( \nabla^2 \frac{1}{r} = -4\pi\delta^3(\mathbf{r}) \).
Also, see Eq. 4.13 in the quantum text for the radial portion of the Laplacian in spherical coordinates; it’s not \( \frac{\partial^2}{\partial r^2} \).

11.9 HINT: You can forget \( \psi_o(\mathbf{r}) \). Also, you will find that \( k = i/a \). In performing the integral \( \int \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} V(\mathbf{r}') \psi(\mathbf{r}') d^3\mathbf{r}' \), you may align the coordinate such that \( \hat{z}' \) aligns with \( \mathbf{r} \).

Then \( \mathbf{r} = r\hat{z}' \), while \( \mathbf{r}' = \mathbf{x}'(\sin \theta' \cos \phi' + \sin \theta' \sin \phi' + \cos \theta') \). Then
\[
|\mathbf{r}'-\mathbf{r}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}
\]
reduces to \( \sqrt{r^2 + r'^2 - 2rr'\cos \theta'} \). The integration over \( \phi' \) is trivial. Next, perform the integration over \( \theta' \). Remarkably, the derivative of the exponent is present, so that integral is actually easy:
\[
\frac{\pi}{\int_0^\infty d\theta' \sin \theta'} \frac{e^{-\frac{1}{a}\sqrt{r^2+r'^2-2rr'\cos \theta'}}}{\sqrt{r^2 + r'^2 - 2rr'\cos \theta'}} = -\frac{a}{rr'} e^{\frac{1}{a}\sqrt{r^2+r'^2-2rr'\cos \theta'}} \bigg|_0^\infty = -\frac{a}{rr'} \left[ e^{-\frac{1}{a}(r+r')} - e^{-\frac{1}{a}|r-r'|} \right]
\]
The final integration is also easy, but watch out for the absolute value sign in the one exponent. That integration has to be split into two pieces: \( \int_0^r + \int_r^\infty \).