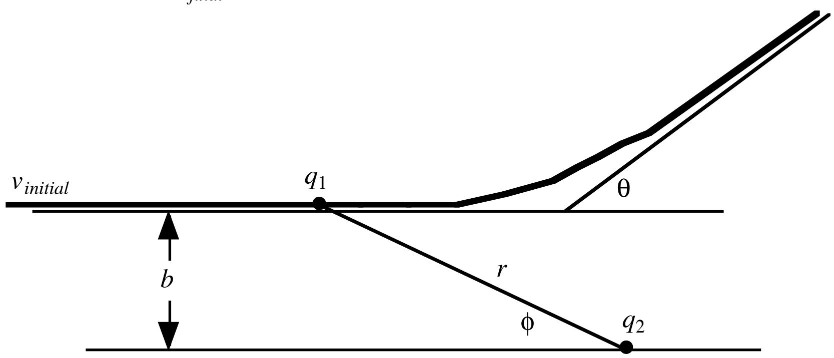
Physics 452: Homework #17

Due Tuesday, Mar. 17, 5:00PM, 2009

11.1 BIG HINT: To solve for the trajectory in part (a), use coordinates r and ϕ.
Note: $\theta = \pi - \phi_{final}$.



Energy $E = \frac{1}{2} m v_{initial}^2$ an angular momentum are conserved: $L = m v_{initial} b = b\sqrt{2mE}$.
To solve for the trajectory, use

$E = T + V \Rightarrow E = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$ Note: $\dot{r} = \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \dot{\phi}\frac{dr}{d\phi}$ and $L = mr^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{L}{mr^2}$.

$\Rightarrow E = \frac{1}{2} m\left(\frac{L}{mr^2}\right)^2\left[\left(\frac{dr}{d\phi}\right)^2 + r^2\right] + V(r)$, from which obtain $\frac{dr}{d\phi} = \pm\frac{r}{b}\sqrt{r^2 - \frac{q_1 q_2}{4\pi\varepsilon_o E} r - b^2}$.

As the particle is approaching, we have $\frac{dr}{d\phi} < 0$, and as it leaves, we have $\frac{dr}{d\phi} > 0$. The sign switches when q_1 is closest to q_2. Therefore, $\left.\frac{dr}{d\phi}\right|_{r=r_{\min}} = 0$ or

$\frac{r_{\min}}{b}\sqrt{r_{\min}^2 - \frac{q_1 q_2}{4\pi\varepsilon_o E} r_{\min} - b^2} = 0$. The only sensible root is $r_{\min} = \frac{-\beta + \sqrt{\beta^2 + 4b^2}}{2} > 0$, where

$\beta \equiv -\frac{q_1 q_2}{4\pi\varepsilon_o E}$. Now solving for the final trajectory,

$$\phi_{final} = \int_{\infty}^{r_{\min}} \frac{dr}{\left(-\frac{r}{b}\sqrt{r^2 - \frac{q_1 q_2}{4\pi\varepsilon_o E} r - b^2}\right)} + \int_{r_{\min}}^{\infty} \frac{dr}{\left(+\frac{r}{b}\sqrt{r^2 - \frac{q_1 q_2}{4\pi\varepsilon_o E} r - b^2}\right)} = 2b\int_{r_{\min}}^{\infty} \frac{dr}{r\sqrt{r^2 - \frac{q_1 q_2}{4\pi\varepsilon_o E} r - b^2}}.$$

Useful: $\int \frac{dx}{x\sqrt{\alpha x^2 + \beta x + \gamma}} = \frac{1}{\sqrt{-\gamma}}\sin^{-1}\left(\frac{\beta x + 2\gamma}{|x|\sqrt{\beta^2 - 4\alpha\gamma}}\right)$. When the integrated expression is evaluated at $r_{\min}$, you should find that the argument of $\sin^{-1}$ becomes one.

See problems on next page.

11.2

11.8 HINT: Eq. 1.102 in your E&M text (Physics 441-442): $\nabla^2 \frac{1}{r} = -4\pi\delta^3(\vec{r})$.
Also, see Eq. 4.13 in the quantum text for the radial portion of the Laplacian in spherical coordinates; it's not $\frac{\partial^2}{\partial r^2}$.

11.9 HINT: You can forget $\psi_o(\vec{r})$. Also, you will find that $k = i/a$. In performing the integral $\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(r')\psi(r')d^3r'$, you may align the coordinate such that $\hat{z}'$ aligns with $\vec{r}$. Then $\vec{r} = r\hat{z}'$, while $\vec{r}' = \hat{x}'\sin\theta'\cos\phi' + \hat{y}'\sin\theta'\sin\phi' + \hat{z}'\cos\theta'$. Then $|\vec{r}-\vec{r}'| \equiv \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ reduces to $\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}$. The integration over ϕ' is trivial. Next, perform the integration over θ'. Remarkably, the derivative of the exponent is present, so that integral is actually easy:

$$\int_0^{\pi} d\theta' \sin\theta' \frac{e^{-\frac{1}{a}\sqrt{r^2+r'^2-2rr'\cos\theta'}}}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} = -\frac{a}{rr'} e^{-\frac{1}{a}\sqrt{r^2+r'^2-2rr'\cos\theta'}}\Bigg|_0^{\pi} = -\frac{a}{rr'}\left[e^{-\frac{1}{a}(r+r')} - e^{-\frac{1}{a}|r-r'|}\right]$$

The final integration is also easy, but watch out for the absolute value sign in the one exponent. That integration has to be split into two pieces: $\int_0^r + \int_r^{\infty}$.