Physics 451: Homework #22  
Due Tuesday, Dec. 9, 5:00PM, 2008

12.1

4.50 HINT: Let $S_z^{(1)} = S_z^{(2)}$ and $S_r^{(1)} = S_r^{(2)}$ (see problem 4.30), so $\hat{a} = \hat{z}$ and $\hat{b} = \hat{r}$. Write the state as $\chi = \frac{1}{\sqrt{2}} (\chi_+^{(1)} \chi_-^{(2)} - \chi_-^{(1)} \chi_+^{(2)})$, where we use an arbitrary spinor $\chi_+ = (\alpha \beta^*)$, which is not necessarily an eigen vector of either operator. We require $\langle \chi_- | \chi_+ \rangle = \chi_+^\dagger \chi_+ = 0$, so

$\chi_- = \begin{pmatrix} \beta^* \\ -\alpha^* \end{pmatrix}$. Also note that $|\alpha|^2 + |\beta|^2 = 1$. Remember don’t mingle different superscripts:

$\langle S_z^{(1)} S_r^{(2)} \rangle = \frac{1}{\sqrt{2}} (\chi_+^{(1)} \chi_-^{(2)}) \langle S_z^{(2)} S_r^{(2)} \rangle \frac{1}{\sqrt{2}} (\chi_+^{(1)} \chi_-^{(2)})$

$= \frac{1}{2} \left[ \langle S_z^{(1)} | S_z^{(2)} \rangle \langle S_r^{(2)} | S_r^{(2)} \rangle - \langle S_z^{(1)} | S_r^{(2)} \rangle \langle S_r^{(2)} | S_z^{(2)} \rangle + \langle S_z^{(1)} | S_z^{(2)} \rangle \langle S_r^{(2)} | S_r^{(2)} \rangle \right]$

12.peatross1 Reproduce Bell’s theorem with your book closed. Grade yourself on how well you know and understand the theorem. Write 100% for full credit.

12.peatross2 Adopt the following local (and therefore incorrect) model of spin: Let $\hat{r}$ be the direction of a particle’s spin (a hidden variable). Let a detector oriented along $\hat{a}$ assign spin up if $\hat{r} \cdot \hat{a} > 0$, and let it assign spin down if $\hat{r} \cdot \hat{a} < 0$. Now consider two such detectors aligned along $\hat{a}$ and $\hat{b}$, and let a series of opposite-spin particle pairs, aligned along $\hat{r}$ and $-\hat{r}$, be measured by the detectors (one particle from each pair measured by each detector). In this model, the particles ‘know’ their orientations, $\hat{r}$ and $-\hat{r}$ (the hidden variable). For simplicity, let $\hat{r}$ orient randomly in the plane containing $\hat{a}$ and $\hat{b}$.

1. Let $\phi$ be the angle between $\hat{r}$ and $\hat{a}$

\[\hat{a}, \hat{b}\]

(a) Find the average product of the two measurements $P(\hat{a}, \hat{b})$ (in units where an up measurement by a detector is represented by +1 and a down measurement is represented by −1). You may express your answer as a plot in terms of the angle $\theta$ between $\hat{a}$ and $\hat{b}$.

(b) Check that this local model obeys Bell’s inequality by picking an arbitrary (non-trivial) angle $\theta$. 

\[\text{spin oriented along arbitrary } \hat{r} \text{ (the hidden variable)} \]
SOLUTION TO 2(a):

\[
A(\hat{a}, \hat{r}) = \begin{cases} 
+1 & \text{if } \hat{r} \cdot \hat{a} > 0 \\
-1 & \text{if } \hat{r} \cdot \hat{a} < 0 
\end{cases}
\]

and

\[
B(\hat{b}, \hat{r}) = \begin{cases} 
-1 & \text{if } \hat{r} \cdot \hat{b} > 0 \\
+1 & \text{if } \hat{r} \cdot \hat{b} < 0 
\end{cases}
\]

where the standard Heaviside function is given by

\[
H(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x < 0 
\end{cases}
\]

Let the angle between \(\hat{b}\) and \(\hat{a}\) be \(\theta\), and let the angle between \(\hat{r}\) and \(\hat{a}\) be \(\phi\). Then, everything is referenced to \(\hat{a}\). The hidden variable is \(\phi\), designating the spin angles (i.e., \(\cos \phi = \hat{a} \cdot \hat{r}\)). \(\theta\) signifies the relative orientation of the detectors (i.e., \(\cos \theta = \hat{a} \cdot \hat{b}\)).

The integral over the hidden variable (which gives \(P\)) may be viewed as an average of the product of \(A\) and \(B\). (Recall that we assume a random distribution of spin orientation \(\phi\), which gives a uniform \(\rho(\phi)\).)

\[
P(\hat{a}, \hat{b}) = \left\langle A(\hat{a} \cdot \hat{r}) B(\hat{b} \cdot \hat{r}) \right\rangle_\phi = -4 \left\langle \left( H(\cos \phi) - \frac{1}{2} \right) \left( H(\cos(\phi - \theta)) - \frac{1}{2} \right) \right\rangle_\phi
\]

\[
= -4 \left[ \left( H(\cos \phi) H(\cos(\phi - \theta)) \right)_\phi - \frac{1}{2} \left( H(\cos(\phi - \theta)) \right)_\phi - \frac{1}{2} \left( H(\cos \phi) \right)_\phi + 1/4 \right]
\]

\[
= -4 \left[ \left( H(\cos \phi) H(\cos(\phi - \theta)) \right)_\phi - \frac{1}{2} (1/2) - \frac{1}{2} (1/2) + 1/4 \right] = 1 - 4 \left( H(\cos \phi) H(\cos(\phi - \theta)) \right)_\phi
\]

By graphing a few points, we determine that our local model gives

We deduce the formula \(P(\hat{a}, \hat{b}) = \frac{2|\theta|}{\pi} - 1\) \((\pi \leq \theta \leq \pi)\), where \(\theta = \cos^{-1}(\hat{a} \cdot \hat{b})\). This should obey Bell’s inequality as will be checked in part (b) by trying a few detector orientations.