Lab 9

LRC Circuits

In this lab you will study AC gain and impedance in LRC circuits.

9.1 LRC Resonance

The series LRC oscillator (see Fig. 9.1) is an important case that has myriad applications. Driven by an AC voltage source, the oscillating current carries energy back and forth between the capacitor (electric field energy) and the inductor (magnetic field energy). The only element where energy is permanently lost is the resistor.

As seen in (7.3), the total impedance of the series circuit is

\[ Z = Z_R + Z_L + Z_C = R + \frac{1}{i\omega C} + i\omega L. \]

The current in the circuit is

\[ I = \frac{V}{Z} = I_0 e^{-i\phi} e^{i\omega t}, \]

which is out of phase with the driving voltage. From example 7.1, the voltage across the resistor compared to the input voltage is

\[ G \equiv \frac{V_R}{V} = \frac{IR}{IZ} = \frac{R}{Z} = |G| e^{-i\phi}. \]  \hspace{1cm} (9.1)

where

\[ |G| = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \]  \hspace{1cm} \text{and} \hspace{0.5cm} \phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right). \] \hspace{1cm} (9.2)

Fig. 9.2 shows the magnitude of the LRC gain $|G|$ as a function of frequency. At resonance, the gain has a magnitude of 1 and phase of zero. At all other frequencies, the magnitude of the gain is less than 1. The peak of the gain curve occurs at the resonance frequency. By inspection of (9.2), the maximum of $|G|$ is achieved when \( \omega L - 1/\omega C = 0 \), which gives a resonance frequency of

\[ \omega_0 = 2\pi f_0 = 1/\sqrt{LC}. \] \hspace{1cm} (9.3)

At the resonance frequency \( \omega = \omega_0 \), it is as if there is no capacitor or inductor in the circuit, so that the impedance becomes purely resistive (R) and the gain is \( G = 1 \). If \( R = 0 \), then the current amplitude at resonance will either 'run away' to...
infinity, or more realistically, the circuit will fail in a spectacular display of light and heat.

A good measure of the width of the resonance peak is obtained by setting $|G| = 1/\sqrt{2}$ and solving for frequency, which yields $\omega \approx \omega_0 \pm \frac{R}{2\pi L}$. Thus, the width is approximately $\Delta \omega \approx \frac{R}{L}$ or $\Delta f \approx \frac{R}{2\pi L}$. The resistance broadens the peak without changing its location.

Many physical systems besides circuits exhibit resonance effects. The crystal shattered by an opera singer, the tall building that sways back and forth during an earthquake, the rattling of a car driving over the top of the 'wake-up' grooves cut into the shoulder of an interstate highway – these are all examples of damped driven resonance. Clearly a shattering crystal or a collapsing building could use some more damping to limit the amplitude of oscillations. Unless strategically damped, most dynamical systems will have in common that they respond strongly to certain resonance frequencies.

### 9.2 RMS Voltage and Current

Since AC voltage and current oscillate, their individual amplitudes time-average to zero. Instead of using an average, it is common to characterize the effective strength of voltage and current by their root-mean-square or rms values. These are written

\[
V_{\text{rms}} \equiv \sqrt{\text{Re}\{V\}^2} = \sqrt{V_0^2 \cos^2 \omega t} = \frac{V_0}{\sqrt{2}}
\]

and

\[
I_{\text{rms}} \equiv \sqrt{\text{Re}\{I\}^2} = \sqrt{I_0^2 \cos^2 (\omega t - \phi)} = \frac{I_0}{\sqrt{2}}
\]

Though $\cos \omega t$ time averages to zero, observe that $\cos^2 \omega t$ is always positive and averages to 1/2. Fig. 9.3 shows the relationship between voltage amplitude $V_0$, peak-to-peak voltage $V_{pp}$, and rms voltage $V_{\text{rms}}$.

![Figure 9.3 AC voltage.](image)

### 9.3 Power

Voltage has units of energy per charge (i.e. J/coul). The voltage specifies the amount of energy required to move charge between different regions of electric potentials. For example, it takes 1.5 J to move a coulomb of charge from one terminal of a 1.5 V battery to the other. Meanwhile, the current flowing in a circuit has units of charge per time (i.e. coul/s). Voltage multiplied by current gives the energy per time (i.e. J/s) or power associated with the motion of charge in a circuit.

Because only the real part of our complex expressions for voltage and current is physical, it is crucial to take the real parts of $V$ and $I$ before multiplying. We compute the power as follows:

\[
P = \text{Re}\{V\} \text{Re}\{I\} = \text{Re}\{V_0 e^{i\omega t}\} \text{Re}\{I_0 e^{i\omega t - i\phi}\}
\]
Eq. (6.10) and (6.19) are handy for taking and manipulating the real parts:

\[ P = V_0 e^{i\omega t} + e^{-i\omega t} I_0 \]

\[ = \frac{V_0 I_0}{2} \left( e^{2i\omega t - i\phi} + e^{-2i\omega t + i\phi} \right) \]

\[ = \frac{V_0 I_0}{2} \left( \cos(2\omega t - \phi) + \cos \phi \right) \quad (9.7) \]

Since the power in an AC circuit fluctuates in time, it is helpful to compute a time average. The first term in (9.7) oscillates positive and negative and so averages to zero. Therefore, the time average of the power is

\[ \bar{P} = \frac{V_0 I_0}{2} \cos \phi = V_{\text{rms}} I_{\text{rms}} \cos \phi \quad (9.8) \]

where in the latter expression we have utilized (9.4) and (9.5).

The factor \( \cos \phi \) is called the power factor. Whereas in a DC circuit the power is simply \( P = V_0 I_0 \), in an AC circuit the phase between the voltage and current matters. Since \( \tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R} \) (see (9.2)), we deduce from Fig. 9.4 that

\[ \cos \phi = \frac{R}{\sqrt{R^2 + \left( \frac{\omega L - \frac{1}{\omega C}}{R} \right)^2}} = \frac{R}{|Z|} \quad (9.9) \]

Inductors and capacitors do not dissipate energy. They can only momentarily store energy. Only a resistor can dissipate energy.

**Example 9.1**

Show using a series LRC circuit that all of the power (9.8) consumed by the circuit is dissipated in the resistor.

**Solution:** Ohm’s law states \( I = \frac{V}{Z} \), which implies \( I_0 = \frac{V_0}{|Z|} \) and \( I_{\text{rms}} = \frac{V_{\text{rms}}}{|Z|} \). Using this and (9.9), the time-averaged power for the circuit (9.8) may be written

\[ \bar{P} = V_{\text{rms}} I_{\text{rms}} \cos \phi = V_{\text{rms}} I_{\text{rms}} \frac{R}{|Z|} = I_{\text{rms}}^2 R \]

On the other hand, the time-averaged power used in the resister is

\[ \bar{P}_R = \text{Re}\{V_R\} \text{Re}\{|I|\} = (\text{Re}\{|I|\})^2 R = I_0^2 R \cos^2 (\omega t - \phi) = \frac{I_0^2 R}{2} = I_{\text{rms}}^2 R \]

where we have used the fact that the cosine squared averages to 1/2. Thus, \( \bar{P}_R = \bar{P} \), meaning all of the power is dissipated in the resister.

Below we list several distinct but equivalent expressions for average power, any one of which might be the most convenient depending which circuit parameters are known.

\[ \bar{P} = I_{\text{rms}}^2 R = I_{\text{rms}}^2 |Z| \cos \phi = I_{\text{rms}} V_{\text{rms}} \cos \phi = \frac{V_{\text{rms}}^2}{|Z|} \cos \phi = \frac{V_{\text{rms}}^2}{R} \cos^2 \phi \quad (9.10) \]
9.4 Parallel and Series Circuits

As was mentioned in Lab 7, AC impedances follow the same parallel and series addition rules as resistance does for DC circuits.

\[ Z_{\text{series}} = Z_1 + Z_2 \quad \text{and} \quad 1/Z_{\text{parallel}} = 1/Z_1 + 1/Z_2 \]

**Example 9.2**

Find the impedance of the circuit depicted in Fig. 9.5.

**Solution:** The impedance of the parallel-branch portion of the circuit is

\[ Z_{\text{parallel}} = \frac{1}{i \omega L} + \frac{1}{1/(i \omega C)} \Rightarrow Z_{\text{parallel}} = \frac{i}{\omega L} - \omega C \]

This impedance is added in series with the resistor, so that

\[ Z = \frac{i}{\omega L} - \omega C + R = \sqrt{R^2 + \frac{1}{(\omega L - \omega C)^2}} e^{i \tan^{-1} \left( \frac{1}{\omega L - \omega C} \right)} \]

The resulting gain \(|G| = R/|Z|\) now has a minimum (rather than a maximum) at \(\omega = 1/\sqrt{LC}\). This circuit is called a *notch filter* because of the ‘hole’ or ‘notch’ in the spectrum near \(\omega_0\). In contrast to the series LRC circuit, the width of the notch is \(\Delta \omega \equiv \frac{1}{RC}\) or \(\Delta f \equiv \frac{1}{2\pi RC}\).

9.5 Equipment

LRC component board, LC meter, signal generator/oscilloscope stack, frequency meter, amplifier, cables.

**Appendix 9.A Time Response of an LRC circuit**

Consider an LRC circuit as depicted in Fig. 9.1. Fig. 9.6 illustrates what happens when the voltage across an LRC circuit is suddenly turned off. From (6.6), the voltage across the circuit is

\[ V = RI + \frac{1}{C} \int I dt + L \frac{dI}{dt} \quad (9.11) \]

If a certain current \(I_0\) is flowing when the voltage is suddenly switched off at \(t = 0\), the equation governing the current thereafter is

\[ \frac{1}{C} \int I dt + L \frac{dI}{dt} = -RI \]

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with solution\(^1\)

\[
I = I_0 e^{-\frac{R}{2L} t} e^{j \sqrt{\frac{R}{LC} - \left(\frac{R}{2L}\right)^2}} \tag{9.12}
\]

The voltage across the resistor is

\[
V_R = IR = R I_0 e^{-\frac{R}{2L} t} e^{j \sqrt{\frac{R}{LC} - \left(\frac{R}{2L}\right)^2}},
\]

which both oscillates\(^2\) and decays in time as shown in Fig. 9.6.

\(^1\)Another valid solution is \(I = I_0 e^{-\frac{R}{2L} t} e^{-j \sqrt{\frac{R}{LC} - \left(\frac{R}{2L}\right)^2}}\), which can be added in certain combinations with (9.11) to produce various other solutions with \(I(t) = I_0\).

\(^2\)The solution oscillates only if \(\frac{1}{LC} > \left(\frac{R}{2L}\right)^2\).
Quiz

Q9.1 A cutoff frequency marks the transition between strong and weak current as a function of frequency for
(a) high-pass filters.
(b) low-pass filters.
(c) both (a) and (b).
(d) an LRC circuit.
(e) all of the above.

Q9.2 For the LRC circuit in Fig. 9.1, let $R = 40 \, \Omega$, $C = 50.0 \, \mu F$ and $L = 50 \, mH$ that is driven by an AC voltage source at frequency $f = 160 \, \text{Hz}$. Compute the following quantities:
(a) $|Z|$ (b) $\phi$ (in degrees)
(c) $|G|$ with the output measured across $R$.

Q9.3 For the circuit in the previous exercise, compute
(a) the resonance frequency $f_0$.

(b) $|Z|$ when $f = f_0$.

Q9.4 Explain how the values of $R$, $C$, and $L$ affect the location and width of the resonance peak.
Exercises

A. Measure and analyze the frequency response of an LRC resonator.

We anticipate that the exercises in this section will proceed smoothly and quickly based on your previous experience RC and RL circuits.

L9.1 Repeat exercise L7.3 for the LRC circuit shown in Fig. 9.7, where \( f_0 = \frac{1}{2\pi\sqrt{LC}} \) is the resonance frequency. Try several different values of \( R \), given your choice of \( L \) and \( C \), and observe resulting peak shapes. Be sure to choose an intermediate value that yields a peak that is neither too broad to fit in the range of the measurement nor so narrow that few of your measured frequencies trace out the peak.

L9.2 Repeat exercises L7.4-L7.5 for the LRC circuit.

L9.3 Repeat the measurements of L7.6-L7.7 for the LRC circuit. When you get to the part of the exercise that involves curve fitting, employ a model of the form \( |G| = \frac{A}{\sqrt{1+\left(\frac{2\pi f L}{R}\right)^2-\frac{1}{2\pi f RC}^2}} \), using \( A \) and \( L \) and \( C \) as variables (fix \( R \) to its measured value). Compare the measured and fitted values of \( L \) and \( C \).

L9.4 Using the fitted values of \( L \) and \( C \), compute the expected resonance frequency in Hz, and compare it to the location of the observed resonance peak.

L9.5 Briefly look through the whole frequency range again to observe the behavior of the phase of the output voltage relative to the input voltage and comment. Keep in mind that time runs from left to right across the oscilloscope screen. Note that the phase must stay within the range between –90° and +90°. A 90° phase corresponds to an input that peaks right where the output is rising through zero. Make a qualitative graph of the frequency-dependent phase for your lab notebook. (You can use \( \phi = 360 f \Delta t \) to verify \( \phi = \pm 90° \) at the endpoints.) Compare with the phase graph predicted in exercise L9.3.

L9.6 (a) Describe how the current in the circuit varies as a function of frequency from one side of the resonance to the other.
(b) What are the magnitudes of the impedance and gain at the resonance frequency? Please comment.
(c) Explain why the phase changes sign at the resonance frequency.

B. Explore a parallel circuit configuration.

L9.7 Reconfigure your circuit to match the notch filter shown in Fig. 9.5 using your same values of \( R \), \( L \), and \( C \). Repeat exercise L9.2, but don't
collect any data. Just scan the frequency range and qualitatively graph the frequency dependence of $|G|$. Try several different values of $R$ and describe its effect on the location and width of the spectral ‘hole’. Refer to example 9.2 as needed.

C. Practice using rms quantities and explore power dissipation.

19.8 Send a sine-wave voltage with amplitude $\sim 2.5$ V (i.e. 5 V peak to peak) from the signal generator (with no DC bias) into the input of the power amplifier. Use the gain control knob to vary the amplitude of the output signal between $V_0 = 0$ and $V_0 \approx 10$ V (i.e. 7 V rms, 20 V peak to peak) while monitoring the output with your oscilloscope to make sure that the voltage oscillates without clipping. Use (9.10) to estimate the rms voltage needed to deliver an average of 1/4 W of power to a 100 $\Omega$ resistor. Then attach a 100 $\Omega$ resistor rated for 1/4 W and slowly turn up the voltage to this level. Does the resistor get hot? Be careful not to burn your finger! If not, continue slowly to turn up the voltage until the resistor is barely too hot to touch without pain, but not much hotter. Record $V_{\text{rms}}$, $I_{\text{rms}}$, and $P$ at this setting. Is the 1/4 W rating appropriate?

![Figure 9.8](image)

**Figure 9.8** 39 k$\Omega$ resistor with 10% uncertainty in its value (Orange - 3, White - 9, Yellow - 10000, Silver - 10%).

![Figure 9.9](image)

**Figure 9.9** Resistor color codes. The power rating is determined by the size of the resistor – not marked except on the package. The most common resistors (small) are rated 1/4 W.